ON THE REGULARIZATION MECHANISM FOR THE PERIODIC KORTEWEG–DE VRIES EQUATION

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This paper is dedicated to the memory of Professor Basil Nicolaenko

Abstract. In this paper we develop and use successive averaging methods for explaining the regularization mechanism in the the periodic Korteweg–de Vries (KdV) equation in the homogeneous Sobolev spaces $\dot{H}^s$, for $s \geq 0$. Specifically, we prove the global existence, uniqueness, and Lipschitz continuous dependence on the initial data of the solutions of the periodic KdV. For the case where the initial data is in L_2 we also show the Lipschitz continuous dependence of these solutions with respect to the initial data as maps from $\dot{H}^s$ to $\dot{H}^s$, for $s \in (-1, 0]$.

Keywords: Korteweg–de Vries equation, dispersive equations, averaging method, nonlinear resonance.

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1. Introduction

This paper is motivated by works on global regularity of solutions of 3D problems in hydrodynamics (Navier-Stokes or Boussinesq system with periodic boundary conditions) in the presence of fast rotation or strong stratification, see, e.g., [1], [2], [3], [4], [11], [12], [18], [16], [20], [26], and references therein. The results are based on the presence of high-frequency waves that lead to destructive interference and effectively weaken the nonlinearity through time averaging allowing to prove global regularity of solutions for these problems, at the limit of infinite rotation. The above mentioned hydrodynamical problems are rather complicated (Boussinesq system, for example, involves five unknown functions of three spatial variables and time, the equations are coupled through a quadratic nonlinearity). Therefore, the purpose of this paper is to apply this averaging approach to a slightly simpler problem to make these ideas more transparent. Here we consider the Korteweg–de Vries equation with one spatial variable subjected to periodic boundary condition. The averaging effects are strong because on high Fourier modes the linear term generates high-frequency oscillations which weakens the nonlinearity and makes it milder. In this paper we show connections between the smoothness properties of solutions of the Korteweg–de Vries (KdV) equation and the algebraic structure of the nonlinear resonances between the high-frequency oscillations. Our main goal is to make the relations between time-averaging effects and smoothness issues more explicit, rather than to obtain global regularity results under minimal restrictions (see, e.g., [5], [7], [14] and the references therein). In particular, our approach and aim are completely different than the machinery and harmonic analysis

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tools that were developed over the past decade and half for investigating dispersive partial differential equations (see the discussion and references mentioned below). Moreover, we also remark that the tools and ideas that are developed in this contribution can be easily applied to multi-dimensional equations and multi-component systems, a subject of future investigation.

It is worth mentioning that similar approach, taking advantage of time averaging, has also been applied to the two-dimensional rotating inviscid Burgers equations in [1] and [16] to show that fast rotation prevents the formation of a finite time singularity in the two-dimensional Burgers system. We will use this warmup toy model, and in particular the approach in [1], to fix ideas and to illustrate in section 2 the averaging mechanism, which is instrumental in regularizing the solutions and in preventing the formation of a finite time singularity for the two-dimensional inviscid Burgers system.

Our approach consists of three steps. First, we rewrite the problem as a system of ODE for time-dependent Fourier coefficients. This is a standard approach to the periodic problem which is commonly used. Second, to make the effects of time averaging explicit we single out oscillating factors and do several integration by parts, with respect to time, to obtain several generations of equations for slowly varying coefficients. Resonances reveal themselves as obstacles to the integrations by parts and produce resonant terms in the equations, integrated terms become more and more regular. Higher generations of equations allow solutions with less regularity. This algorithm is simple and straightforward, it does not use fine methods such as analytic reduction to linear forms and analysis of self similar solutions or the construction of conserved or almost conserved quantities. The third step is analysis of the obtained equations. To show their regularity we use straightforward estimates of multilinear operators, energy estimates and the Contraction principle. To use the contraction principle in negative Sobolev spaces we use splitting to high and low Fourier modes and exploit averaging-induced squeezing of higher modes. In order to justify our estimates we use a Galerkin approximations procedure. We try to use methods which can be easily adjusted to general multidimensional problems such as those mentioned above in hydrodynamic problems. We do not use in our analysis the specific properties of the KdV equation such as complete integrability or special conserved quantities. Moreover, one can expect that in order to obtain more delicate information one has to resort to the use of higher generations of equations, i.e. more integrations by part.

The paper is organized as follows. In section 3 we reformulate the KdV equation in terms of its Fourier components and introduce the notion of weak solution; we also make two types of formal transformations of the equation which will be used for proving the relevant well-posedness results. In section 4 we use the Galerkin procedure for proving the global existence in the Sobolev spaces $\dot{H}^s$ for $s > 0$. In section 5 we use the Contraction Principle to show that the solutions established in section 4 are unique and Lipschitz continuous, with respect to the initial data, in the Sobolev spaces $\dot{H}^s$ for $s \geq 1/2$. A more technical section 6 is devoted to proving similar results for $s \in [0, 1/2]$, where the case $s = 0$ is dealt with separately. Finally, for solutions with initial data in L_2 we show the Lipschitz continuous dependence of these solutions, with respect to the initial data, as maps from $\dot{H}^s$ to $\dot{H}^s$ for all $s \in (-1, 0]$.

There is a long history of the KdV equations in the periodical setting. We do not attempt to give here a complete survey of this long history, but we are satisfied by mentioning, below, some of the key landmark results. First Bourgain [6] proved local well-posedness in L_2. We observe that the result of Bourgain also implies global existence for the real valued KdV in L_2. Later the local well-posedness in the Sobolev space H^s, for $s > -1/2$, was

established for the real and complex valued KdV by Kenig, Ponce, and Vega [14]. Later global well-posedness in the Sobolev space H^s, for $s > -1/2$, was established for the real valued KdV by Colliander, Keel, Staffilani, Takaoka and Tao [7]. Using the fact that the KdV equation is an integrable system, and by implementing the inverse method, Kappeler and Topalov [15] proved global well-posedness in the Sobolev space H^s, for $s > -1$.

Even though the results presented in this contribution are weaker than the state-of-the-art results mentioned above, our approach is substantially different and easy to master. Specifically, and as we have stressed above, our approach is based on systematic application of time averaging. It is worth mentioning that since we rewrite the KdV equation in different forms, based on the differentiation by parts with respect to the time variable, there is superficial similarity with the reduction to normal forms (cf. [19]). The difference is that the consecutive forms of the KdV, which we derive, are based not on the geometric properties such as a reduction to a linear form or a reduction to invariant tori, but exclusively on time averaging properties. In order to stress the effects, which are due solely to the time averaging, we do not use energy conservation or modified energy functionals as in the I-method. We use only the L_2 norm conservation to mimic the situation which occurs in the three-dimensional Euler equations and other hydrodynamical systems.

Moreover, we believe that the averaging approach presented here can be pushed further to achieve the best known results concerning the KdV equation. Furthermore, different nonlinear dispersive and wave equations, such as the m-KdV, the Klein-Gordon equation and water wave equations (see for example, [13], [19], [23]-[25], and references therein), can possibly be studied quite similarly, a subject of future research.

2. The Complex Burgers equation with fast rotation — a paradigm for the averaging mechanism

In this section we consider the complex Burgers equation with fast rotation as a warmup toy model to demonstrate the averaging mechanism for preventing the formation of singularity.

The one-dimensional Burgers equation

$$u_t + uu_x = 0$$

is known to develop singularity in finite time. Considering the complex version of this equation:

$$u_t + uu_z = 0, \qquad u(z,0) = \varphi(z), \tag{2.1}$$

where $u \in \mathbb{C}$, and the initial value, $\varphi(z)$, is a bounded complex analytic function with a bounded derivative in the complex strip $D = \{z \in \mathbb{C} : \ |z| < d\}$. Using the usual characteristic method one can show that the solution of (2.1) is given by the implicit relation:

$$u(z,t) = \varphi(z - tu(z,t)).$$

It is easy to show that the above relation defines a unique complex analytic solution $u(z,t)$ in a smaller sub-strip of D, for small values of $|t|$, depending on the $\sup_{z\in D}|\varphi(z)|$ and $\sup_{z\in D}|\varphi'(z)|$. Moreover, the solution of (2.1) satisfies:

$$\partial_z u = \varphi'(z - tu(z,t))\left(1 + t\varphi'(z - tu(z,t))\right)^{-1}. \tag{2.2}$$

Equation (2.2) shows that for a large class of analytic initial data, φ, the solution of (2.1) develops a singularity in a finite time. In particular, and as for the case of the real Burgers equation, the solution develops a finite time singularity if the initial data, φ, maps $\mathbb{R}$ into $\mathbb{R}$ and is a monotonic decreasing function in some interval of $\mathbb{R}$. Moreover, it also follows

from (2.2) and the Picard little theorem for entire functions that if the initial data φ is an entire function then the solution of (2.1) instantaneously ceases to be an entire function, unless $\varphi'(z) = constant$, i.e., $\varphi(z) = az + b$ for some constants $a, b \in \mathbb{C}$.

Adding a complex rotation term to (2.1) one obtains the rotating complex Burgers equation

$$u_t + uu_z = i\Omega u, \qquad u(z,0) = \varphi(z), \tag{2.3}$$

in the strip D, with rotation rate $\Omega \in \mathbb{R}$. Making the change of variables $v = e^{-i\Omega t}u$ gives the equivalent equation

$$v_t + e^{i\Omega t} v v_z = 0, \qquad v(z,0) = \varphi(z), \tag{2.4}$$

which can also be solved by the characteristic method to obtain the solution in the implicit form

$$v(z,t) = \varphi(z - \frac{e^{i\Omega t} - 1}{i\Omega} v(z,t)). \tag{2.5}$$

Equation (2.5) has a unique complex analytic solution, with respect to the spatial complex variable z, for small values of $|t|$ and in a fixed smaller sub-strip of D, whose size depends on $|\Omega|$, $\sup_{z\in D} |\varphi(z)|$ and $\sup_{z\in D} |\varphi'(z)|$. Observe that

$$\partial_z v(z,t) = \varphi'(z - \frac{e^{i\Omega t} - 1}{i\Omega} v(z,t)) \left(1 + \frac{e^{i\Omega t} - 1}{i\Omega} \varphi'(z - \frac{e^{i\Omega t} - 1}{i\Omega} v(z,t))\right)^{-1}. \tag{2.6}$$

As a result, it is clear that if we choose $|\Omega|$ large enough, such that

$$|\Omega| \geq 2 \sup_{z\in D} |\varphi'(z)|,$$

then $|\partial_z v(z,t)|$ remains finite for all $t \in \mathbb{R}$, and the solution exists globally in time. Consequently, we have just demonstrated that for fast rotation, which depends on the initial data, the unique solution remains regular in a fixed sub-strip of D, for all $t \in \mathbb{R}$. It is worth observing, however, that, again, by virtue of (2.6) and the Picard little theorem the solution of (2.3) instantaneously ceases to be an entire function even if the initial data φ is an entire function, unless $\varphi(z) = az + b$, for some constants $a, b \in \mathbb{C}$.

Observe that equation (2.3) can be rewritten in the integrated form:

$$v(z,t) = \varphi(z) - \int_0^t e^{i\Omega\tau} v(z,\tau)\partial_z v(z,\tau) d\tau.$$

This form exhibits the role that is played by the time averaging process in prolonging the life-span of the solutions for this kind of equations. Indeed, the averaging against the fast oscillating term $e^{i\Omega t}$, for $|\Omega|$ large enough, reduces the "strength" of the nonlinear term and makes it milder, which is the underlined mechanism for prolonging the life-span of the solutions.

Setting $u = u_1 + iu_2$ and $z = x + iy$, we observe that, thanks to the Cauchy-Riemann equations, equation (2.3) is equivalent the two-dimensional Burgers equations with rotation, for the vector field $(u_1(x,y), u_2(x,y))$. This is the same system that was investigated in [16]; and similar results, to the ones mentioned above, were proved using completely different tools.

3. Transformations of the Korteweg–de Vries equation

The toy model presented in section 2 is only an illustrative example to demonstrate the underlying main idea of the averaging mechanism. Here, we consider the Korteweg–de

Vries (KdV) equation with space-periodic boundary condition

$$\partial_t u = u\partial_x u + \partial_x^3 u, \qquad u(0,x) = u^0(x), \tag{3.1}$$

where $x \in \mathbb{S}^1 = [0, 2\pi]$ and $u(t,0) = u(t,2\pi)$. Let $u(x,t)$ be a smooth solution of equation (3.1). Writing the nonlinear term as $u\partial_x u = \frac{1}{2}\partial_x(u^2)$ and integrating with respect to x we see that $\int_0^{2\pi} u(t,x)dx = \int_0^{2\pi} u(0,x)$. Therefore, without loss of generality we can (and shall) assume that the initial data and the solution both have zero spatial mean

$$\int_0^{2\pi} u(t,x)dx = \int_0^{2\pi} u^0(x)dx = 0. \tag{3.2}$$

Multiplying (3.1) by u and integrating with respect to x we formally obtain the well-known L_2-norm conservation property:

$$\|u(t)\|_{L_2} = \|u^0\|_{L_2}.$$

We use the Fourier series in x:

$$u(t,x) = \sum_{k\in\mathbb{Z}_0} u_k(t)e^{ikx}, \quad u_k \in \mathbb{C}, \quad u_k(t) = \frac{1}{2\pi}\int_0^{2\pi} u(t,x)e^{-ikx}dx, \tag{3.3}$$

where, according to (3.2) $u_0(t) \equiv 0$, hence the summation is over $\mathbb{Z}_0 = \mathbb{Z} \setminus \{0\}$, and we have, in addition, that $u_{-k} = \bar{u}_k$, since u is real-valued.

It will be convenient below to normalize the L_2 norm so that

$$\|u\|^2_{\dot{H}^0} = \|u\|^2_{L_2} = \frac{1}{2\pi}\int_0^{2\pi} u(x)^2 dx = \sum_{k\in\mathbb{Z}_0} |u_k|^2.$$

Accordingly, the norms in the Sobolev spaces H^s are defined as follows. We set

$$\|u\|^2_{H^s} = |u_0|^2 + \|u\|^2_{\dot{H}^s}, \tag{3.4}$$

where $\|u\|_{\dot{H}^s}$ is the norm in the homogeneous space $\dot{H}^s$ of functions with mean value zero

$$\|u\|^2_{\dot{H}^s} = \|\{|k|^s u_k\}\|^2_{l_2} = \sum_{k\in\mathbb{Z}_0} |k|^{2s}|u_k|^2, \quad s \in \mathbb{R}.$$

We will use the above definition for a norm of a function $u(x)$ as well as for a norm of a sequence $\{u_k\}$.

Using the Fourier representation of u (a smooth solution of (3.1)) we write equation (3.1) as the infinite coupled system of ordinary differential equations for the coefficients $u_k(t)$:

$$\partial_t u_k = \frac{1}{2}ik\sum_{k_1+k_2=k} u_{k_1}u_{k_2} - ik^3 u_k, \quad u_k(0) = u_k^0, \quad k \in \mathbb{Z}_0. \tag{3.5}$$

Next, we use the following transformation of variables

$$u_k(t) = e^{-ik^3 t}v_k(t), \qquad k \in \mathbb{Z}_0. \tag{3.6}$$

Since $v_{-k}(t) \equiv \bar{v}_k(t)$ for all $k \in \mathbb{Z}_0$ then the functions $v(t,x)$ is real-valued along with $u(t,x)$, and for every s the Sobolev norm is preserved:

$$\|u(t)\|_{\dot{H}^s} = \|v(t)\|_{\dot{H}^s}.$$

We observe that this change of variables is similar to the one introduced for (2.3), and has also been used in [2], [3] for proving global regularity of the 3D rotating Navier–Stokes equations, see also [4], [17] and [26]. In the first place, the substitution (3.6) eliminates the linear term ik^3 in (3.5), which has the highest, namely cubic, order of growth as

$|k| \to \infty$ and, most importantly, introduces oscillating exponentials into the nonlinear term. Substituting (3.6) in (3.5), multiplying by e^{ik^3t}, and using the identity

$$(k_1 + k_2)^3 - k_1^3 - k_2^3 = 3(k_1 + k_2)k_1k_2, \tag{3.7}$$

we obtain the equivalent coupled system of equations

$$\partial_t v_k = \frac{1}{2} ik \sum_{k_1+k_2=k} e^{i3kk_1k_2t} v_{k_1} v_{k_2}, \quad v_k(0) = v_k^0 = u_k^0, \quad k \in \mathbb{Z}_0. \tag{3.8}$$

Since our techniques are based on Fourier expansions, we take equation (3.8) as our primary form of the KdV, which is equivalent to the classical KdV equation (3.1) for smooth solutions.

Definition 3.1. A function v is called a solution of (3.8) over the time interval $[0, T]$ if $v \in L_\infty([0, T]; \dot{H}^0)$ and the integrated equation (3.8)

$$v_k(t) - v_k(0) = \frac{1}{2} ik \int_0^t \sum_{k_1+k_2=k} e^{i3kk_1k_2t'} v_{k_1}(t') v_{k_2}(t') dt', \quad k \in \mathbb{Z}_0 \tag{3.9}$$

is satisfied for every $k \in \mathbb{Z}_0$. Accordingly, $u(x, t)$ with $u_k(t) = e^{-ik^3t} v_k(t)$ will then be called a weak solution of the KdV.

Remark 3.1. We observe that if v is a solution in the sense of Definition 3.1, then by setting $d_k(t) = \sum_{k_1+k_2=k} e^{i3kk_1k_2t} v_{k_1}(t) v_{k_2}(t)$ we have

$$\sup_{t\in[0,T]} |d_k(t)| \le \|v\|^2_{L_\infty([0,T];\dot{H}^0)}, \tag{3.10}$$

and hence we automatically get from (3.9) that $v_k(t)$ is an absolutely continuous function over $[0, T]$ for each k, and consequently (3.8) is satisfied a. e. in $[0, T]$. Moreover, by Lemma 7.1 we have $B_1(v, v) \in L_\infty([0, T]; \dot{H}^{-\theta})$, for $\theta > 3/2$, where

$$B_1(\varphi, \psi)_k = \frac{1}{2} ik \sum_{k_1+k_2=k} e^{i3kk_1k_2t} \varphi_{k_1} \psi_{k_2}, \quad k \in \mathbb{Z}_0. \tag{3.11}$$

Therefore, satisfying (3.8) in the sense of Definition 3.1 implies satisfying the functional equation

$$\partial_t v = B_1(v, v), \quad v(0) = v^0, \tag{3.12}$$

in the sense of $L_p([0, T]; \dot{H}^{-\theta})$, where $\theta > 3/2$ and $1 \le p \le \infty$.

Assume we have a smooth enough solution of the KdV equation (3.1), and hence of equation (3.8). We will do all kinds of manipulations to get various forms of the equations. This will be done by differentiation by parts procedure and the repeated use of equation (3.8). Notice that formally the solutions of (3.8) will satisfy the newly derived equations. We will derive these equations formally in order to motivate the use of the corresponding Galerkin truncated versions of these equations, which will play a major role in the rigorous justification of the steps of our proofs.

3.1. ***First differentiation by parts in time.*** Equation (3.8) does not involve ik^3 as in (3.5), but still involves explicitly the factor $\frac{1}{2}ik$, which tends to infinity as $|k| \to \infty$. To obtain a system for the Fourier coefficients that has coefficients uniformly bounded in k we rewrite (3.8) in a different form. The formal transformation, which is valid for smooth enough solutions, corresponds to the integration by parts in (3.9), but since we rewrite

the differential equation (3.8), we call it differentiation by parts. Since $v_0(t) \equiv 0$, we can assume in (3.8) that $k_1, k_2 \neq 0$ so that $\partial_t((i3kk_1k_2)^{-1}e^{i3kk_1k_2t}) = e^{i3kk_1k_2t}$, and therefore

$$\begin{aligned}\partial_t v_k = \partial_t \left(\frac{1}{2}ik \sum_{k_1+k_2=k} \frac{e^{i3kk_1k_2t}v_{k_1}v_{k_2}}{i3kk_1k_2}\right) - \frac{1}{2}ik \sum_{k_1+k_2=k} \frac{e^{i3kk_1k_2t}}{i3kk_1k_2}\partial_t(v_{k_1}v_{k_2}) = \\ \frac{1}{6}\partial_t \left(\sum_{k_1+k_2=k} \frac{e^{i3kk_1k_2t}v_{k_1}v_{k_2}}{k_1k_2}\right) - \frac{1}{6}\sum_{k_1+k_2=k} \frac{e^{i3kk_1k_2t}}{k_1k_2}(v_{k_2}\partial_t v_{k_1} + v_{k_1}\partial_t v_{k_2}).\end{aligned} \tag{3.13}$$

The last two terms are symmetric with respect to k_1 and k_2, and it suffices to consider one of them. For the term containing, say, $v_{k_2}\partial_t v_{k_1}$ we use (3.8) for $\partial_t v_{k_1}$ and obtain

$$\begin{aligned}\sum_{k_1+k_2=k} \frac{e^{i3kk_1k_2t}}{k_1k_2}v_{k_2}\partial_t v_{k_1} = \frac{1}{2}i \sum_{k_1+k_2=k} \frac{e^{i3kk_1k_2t}}{k_2}v_{k_2} \sum_{\alpha+\beta=k_1} e^{i3k_1\alpha\beta t}v_\alpha v_\beta = \\ \frac{1}{2}i \sum_{\alpha+\beta+k_2=k} \frac{e^{i3(k(\alpha+\beta)k_2+(\alpha+\beta)\alpha\beta)t}}{k_2}v_{k_2}v_\alpha v_\beta = \frac{1}{2}i \sum_{k_1+k_2+k_3=k} \frac{e^{i3(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_2}v_{k_1}v_{k_2}v_{k_3},\end{aligned}$$

where we have factorized the exponent as follows

$$kk_2 + \alpha\beta = (\alpha+\beta+k_2)k_2 + \alpha\beta = (k_2+\alpha)(k_2+\beta).$$

Remark 3.2. This is, of course, just the identity

$$(k_1+k_2+k_3)^3 - k_1^3 - k_2^3 - k_3^3 = 3(k_1+k_2)(k_2+k_3)(k_3+k_1). \tag{3.14}$$

We also observe that the cubic identities (3.7) and (3.14) are known to be very important in the analysis of the KdV equation both on $\mathbb{R}$ and in the periodic setting [5], [7].

Since the expression for the term containing $v_{k_1}\partial_t v_{k_2}$ is exactly the same, we finally obtain the following form of the KdV

$$\partial_t \left(v_k - \frac{1}{6}B_2(v,v)_k\right) = \frac{i}{6}R_3(v,v,v)_k, \qquad k \in \mathbb{Z}_0, \tag{3.15}$$

where

$$B_2(u,v)_k = B_2(u,v,t)_k = \sum_{k_1+k_2=k} \frac{e^{i3kk_1k_2t}u_{k_1}v_{k_2}}{k_1k_2}, \qquad k \in \mathbb{Z}_0, \tag{3.16}$$

and

$$R_3(u,v,w)_k = \sum_{k_1+k_2+k_3=k} \frac{e^{i3(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1}u_{k_1}v_{k_2}w_{k_3}, \qquad k \in \mathbb{Z}_0. \tag{3.17}$$

Recall that the transformation between the variables u and v is an isometry between the Sobolev spaces $\dot{H}^s$. Moreover, we will show later that all the terms involved in the above form of the KdV are bounded operators, in the appropriate functional spaces. Consequently, the above form of the KdV is a much milder form than the original forms in equations (3.1) and (3.8). Since we will introduce later another form of this system using one more differentiation by parts, we also call (3.15) *the first form of the KdV*. We integrate (3.15) with respect to t and arrive at the following *integrated first form of the KdV*:

$$v_k(t) - \frac{1}{6}B_2(v(t),v(t))_k = v_k(0) - \frac{1}{6}B_2(v^0,v^0)_k + \frac{i}{6}\int_0^t R_3((v(\tau))^3)_k d\tau, \quad v_k(0) = v_k^0, \tag{3.18}$$

where $R_3(v^3) = R_3(v,v,v)$ is given by (3.17) and $B_2(v,v)$ is given by (3.16).

Remark 3.3. Let v be a solution of (3.8) in the sense of Definition 3.1. Then we observe that in view of Lemma 7.16 (with $\beta = 0$) the trilinear operator R_3 is a bounded map from $(\dot{H}^0)^3$ into $\dot{H}^{-\theta}$, for $\theta > 1/2$, so that equation (3.15) in the functional form holds in $L_\infty([0,T];\dot{H}^{-\theta})$.

Here we will go from (3.15) along two different paths depending on the purpose:

- *a priori* estimates (second differentiation by parts in time);
- uniqueness and Lipschitz continuous dependence of the solutions on the initial data.

3.2. ***Second differentiation by parts in time.*** We prove later in Lemma 7.15 that the trilinear operator $R_3(u,v,w)$ is a bounded map from the Sobolev space $(\dot{H}^s)^3$ into $\dot{H}^s$ for any $s > 1/2$. The bilinear operator $B_2(u,v)$ has nicer continuity properties and is bounded from $(\dot{H}^s)^2$ into $\dot{H}^s$ for any $s > -1/2$, see Lemma 7.2. Therefore, from the continuity properties of R_3 we are unable to use (3.15) to establish the required *a priori* estimates for $s \geq 0$. For this reason we use again the idea of differentiation by parts, and once again represent the exponential in (3.17) as the time derivative. But before doing that we have to take care of the resonances which are the obstruction to the integration by parts procedure.

Resonances. We single out the terms in (3.17) for which

$$(k_1+k_2)(k_2+k_3)(k_3+k_1) = 0, \qquad k_1+k_2+k_3 = k \in \mathbb{Z}_0. \tag{3.19}$$

Accordingly, we have

$$\begin{aligned} &R_3(v,v,v)_k = R_{3\mathrm{res}}(v^3)_k + R_{3\mathrm{nres}}(v^3)_k,\\ &R_{3\mathrm{res}}(v^3)_k = \sum_{k_1+k_2+k_3=k}^{\mathrm{res}} \frac{v_{k_1}v_{k_2}v_{k_3}}{k_1},\\ &R_{3\mathrm{nres}}(v^3)_k = \sum_{k_1+k_2+k_3=k}^{\mathrm{nonres}} \frac{e^{i3(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1} v_{k_1}v_{k_2}v_{k_3}, \end{aligned} \tag{3.20}$$

where the first summation is carried out over the set subscripts k_1, k_2, k_3 satisfying (3.19) (the resonance), while in the second summation $(k_1+k_2)(k_2+k_3)(k_3+k_1) \neq 0$ (the non-resonant terms).

Since it easy to see that not all three factors in (3.19) can be zero at the same time, it follows that the set of k_1, k_2, k_3 satisfying (3.19) is the union of six disjoint sets $S_1, \ldots, S_6$:

$$\begin{aligned} &S_1 = \{k_1+k_2=0\} \cap \{k_2+k_3=0\} \;\Leftrightarrow k_1 = k, k_2 = -k, k_3 = k,\\ &S_2 = \{k_1+k_2=0\} \cap \{k_3+k_1=0\} \;\Leftrightarrow k_1 = -k, k_2 = k, k_3 = k,\\ &S_3 = \{k_2+k_3=0\} \cap \{k_3+k_1=0\} \;\Leftrightarrow k_1 = k, k_2 = k, k_3 = -k,\\ &S_4 = \{k_1+k_2=0\} \cap \{k_2+k_3\neq 0\} \cap \{k_3+k_1\neq 0\} \;\Leftrightarrow k_1 = j, k_2 = -j, k_3 = k, |j| \neq k,\\ &S_5 = \{k_2+k_3=0\} \cap \{k_1+k_2\neq 0\} \cap \{k_3+k_1\neq 0\} \;\Leftrightarrow k_1 = k, k_2 = j, k_3 = -j, |j| \neq k,\\ &S_6 = \{k_3+k_1=0\} \cap \{k_1+k_2\neq 0\} \cap \{k_2+k_3\neq 0\} \;\Leftrightarrow k_1 = j, k_2 = k, k_3 = -j, |j| \neq k, \end{aligned}$$

where $j \in \mathbb{Z}_0$. Therefore,

$$
\begin{aligned}
R_{3\mathrm{res}}(v^3)_k = \sum_{m=1}^{6}\sum_{S_m} \frac{v_{k_1}v_{k_2}v_{k_3}}{k_1} = \frac{v_k v_{-k} v_k}{k} + \frac{v_{-k} v_k v_k}{-k} + \frac{v_k v_k v_{-k}}{k} + \\
v_k \sum_{j\in\mathbb{Z}_0, |j|\neq k} \frac{v_j v_{-j}}{j} + \frac{v_k}{k} \sum_{j\in\mathbb{Z}_0, |j|\neq k} |v_j|^2 + v_k \sum_{j\in\mathbb{Z}_0, |j|\neq k} \frac{v_j v_{-j}}{j} = \\
\frac{v_k}{k}\left(|v_k|^2 + \sum_{j\in\mathbb{Z}_0} |v_j|^2 - |v_k|^2 - |v_{-k}|^2\right) = \frac{v_k}{k}(\|v\|_{L_2}^2 - |v_k|^2) =: A_{\mathrm{res}}(v)_k,
\end{aligned}
\tag{3.21}
$$

where we used the fact that the first two terms add up to zero, the fourth and the sixth terms are both zero by symmetry $j \to -j$.

Remark 3.4. We observe that if the energy is conserved, $\|v(t)\|_{L_2} = \|v(0)\|_{L_2} = \|v^0\|_{L_2} = \|u^0\|_{L_2}$, then clearly

$$
R_{3\mathrm{res}}(v^3)_k = \frac{v_k}{k}(\|v\|_{L_2}^2 - |v_k|^2) = \frac{v_k}{k}(\|v^0\|_{L_2}^2 - |v_k|^2) =: A_{\mathrm{res}}(v)_k. \tag{3.22}
$$

For smooth enough solutions equation (3.15) can now be written in the form

$$
\partial_t \left(v_k - \frac{1}{6}B_2(v,v)_k\right) = \frac{i}{6}A_{\mathrm{res}}(v)_k + \frac{i}{6}R_{3\mathrm{nres}}(v^3). \tag{3.23}
$$

Since the exponent in the last term on the right-hand side does not vanish, we can differentiate by parts with respect to t a second time:

$$
\begin{aligned}
R_{3\mathrm{nres}}(v^3)_k = \sum_{k_1+k_2+k_3=k}^{\mathrm{nonres}} \frac{e^{i3(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1} v_{k_1}v_{k_2}v_{k_3} = \frac{1}{3i}\partial_t B_3(v,v,v)_k - \\
\frac{1}{3i}\sum_{k_1+k_2+k_3=k}^{\mathrm{nonres}} \frac{e^{i3(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1(k_1+k_2)(k_2+k_3)(k_3+k_1)}(\partial_t v_{k_1}v_{k_2}v_{k_3} + v_{k_1}\partial_t v_{k_2}v_{k_3} + v_{k_1}v_{k_2}\partial_t v_{k_3}),
\end{aligned}
\tag{3.24}
$$

where

$$
B_3(u,v,w)_k = \sum_{k_1+k_2+k_3=k}^{\mathrm{nonres}} \frac{e^{i3(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1(k_1+k_2)(k_2+k_3)(k_3+k_1)} u_{k_1}v_{k_2}w_{k_3}. \tag{3.25}
$$

As before, we express the time derivatives in the last term on the right-hand side in (3.24) by means of equation (3.8). The terms containing $\partial_t v_{k_2}$ and $\partial_t v_{k_3}$ produce the same two expressions and after a straight forward calculation we obtain

$$
\begin{aligned}
\sum_{k_1+k_2+k_3=k}^{\mathrm{nonres}} \frac{e^{i3(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1(k_1+k_2)(k_2+k_3)(k_3+k_1)}(\partial_t v_{k_1}v_{k_2}v_{k_3} + v_{k_1}\partial_t v_{k_2}v_{k_3} + v_{k_1}v_{k_2}\partial_t v_{k_3}) = \\
= iB_4(v,v,v,v)_k,
\end{aligned}
\tag{3.26}
$$

where

$$
B_4(u,v,w,\varphi)_k = \frac{1}{2}B_4^1(u,v,w,\varphi)_k + B_4^2(u,v,w,\varphi)_k \tag{3.27}
$$

and the term corresponding to $\partial_t v_{k_1}$ is B_4^1:

$$
B_4^1(u,v,w,\varphi)_k = \sum_{k_1+k_2+k_3+k_4=k}^{\mathrm{nonres}} \frac{e^{i\Phi(k_1,k_2,k_3,k_4)t}}{(k_1+k_2)(k_1+k_3+k_4)(k_2+k_3+k_4)} u_{k_1}v_{k_2}w_{k_3}\varphi_{k_4}, \tag{3.28}
$$

and the sum of the terms corresponding to $\partial_t v_{k_2}$ and $\partial_t v_{k_3}$ is B_4^2:

$$B_4^2(u,v,w,\varphi)_k = \sum_{k_1+k_2+k_3+k_4=k}^{\text{nonres}} \frac{e^{i\Phi(k_1,k_2,k_3,k_4)t}\ (k_3+k_4)}{k_1(k_1+k_2)(k_1+k_3+k_4)(k_2+k_3+k_4)} u_{k_1} v_{k_2} w_{k_3} \varphi_{k_4}. \tag{3.29}$$

The phase function Φ here is

$$\Phi(k_1,k_2,k_3,k_4) = (k_1+k_2+k_3+k_4)^3 - k_1^3 - k_2^3 - k_3^3 - k_4^3.$$

Unlike (3.7) or (3.14), the above phase function does not have a nice factorization in the general case. However, the particular analytic expression for Φ is not used in our subsequent analysis.

Hence, we have for $R_{3\text{nres}}(v^3)_k$:

$$R_{3\text{nres}}(v^3)_k = \frac{1}{3i}\partial_t B_3(v,v,v)_k - \frac{1}{3}\left(\frac{1}{2}B_4^1(v,v,v,v)_k + B_4^2(v,v,v,v)_k\right). \tag{3.30}$$

Combining (3.23)–(3.25), (3.27) and (3.30) we finally formally deduce the following equivalent form of the original equation:

$$\partial_t\left(v_k - \frac{1}{6}B_2(v,v)_k - \frac{1}{18}B_3(v,v,v)_k\right) = \frac{i}{6}A_{\text{res}}(v)_k + \frac{i}{18}B_4(v,v,v,v)_k, \quad k\in\mathbb{Z}_0, \tag{3.31}$$

where B_2 is defined in (3.16), B_3 is defined in (3.25), A_{res} is defined in (3.21), and B_4 is defined in (3.26). We call this equation *the second form of the KdV*. The smoothing properties of these multi-linear operators, involved in (3.31) are established in section 7. The integrated equation of (3.31) takes the fom

$$\begin{aligned}\left(v_k - \frac{1}{6}B_2((v)^2)_k - \frac{1}{18}B_3((v)^3)_k\right)(t) - \left(v_k - \frac{1}{6}B_2((v)^2)_k - \frac{1}{18}B_3((v)^3)_k\right)(0) = \\ = \int_0^t \left(\frac{i}{6}A_{\text{res}}(v)_k(t') + \frac{i}{18}B_4((v)^4)_k(t')\right)dt'.\end{aligned} \tag{3.32}$$

Remark 3.5. Let v be a solution of (3.8) in the sense of Definition 3.1. Then in view of Lemma 7.11 and Lemma 7.14 equation (3.31), as a functional equation, holds in $L_\infty([0,T];\dot{H}^0)$.

3.3. ***Uniqueness and Lipschitz continuous dependence on the initial data***. As will be shown in section 4 for the case of regular initial data, i.e. $v^0\in\dot{H}^s$ for $s>1/2$, one will be able to use the continuity properties of the trilinear operator R_3 to show the uniqueness and Lipschitz continuous dependence on the initial data of the solutions of (3.8) in the sense of Definition 3.1 by the direct use of the equation (3.15).

However, for a less regular class of initial data, i.e. $v^0\in\dot{H}^s$ for $s\in[0,1/2]$ (*hereafter this class is called non-regular*), we are unable to use equation (3.15), as in the case of regular initial data, to show the uniqueness and Lipschitz continuity of the solutions of (3.8). Therefore, we are required to provide a more delicate analysis to show the uniqueness. In particular, we will use certain family of equations for this task which are even more technical to present here and will be the subject of section 6 (see equations (6.19) parameterized by $n\in\mathbb{N}$).

4. Truncated system, *a priori* estimates and global existence for $s>0$.

In this section we establish the global existence (without uniqueness) of solutions of equation (3.8) in the sense of Definition 3.1 for initial data $v^0\in\dot{H}^s$, where $s>0$. Moreover, we will show that the established solutions conserve the energy, namely, they satisfy

$\|v(t)\|_{L_2} = \|v^0\|_{L_2}$. For this purpose we introduce a Galerkin version of equation (3.8), namely we replace (3.8) by an equation, with *truncated* nonlinearity, for the approximate solution $v = v^{(m)}$ for $m \in \mathbb{N}$ (in what follows we omit the superscript (m) where it causes no ambiguity)

$$\partial_t v_k^{(m)} = \frac{1}{2} ik\Pi_m \sum_{k_1+k_2=k} e^{i3kk_1k_2t}(\Pi_m v_{k_1}^{(m)})(\Pi_m v_{k_2}^{(m)}) \qquad v_k^{(m)}(0) = v_k^0 = u_k^0, \tag{4.1}$$

where $k \in \mathbb{Z}_0$. Here the truncation (projection) operator Π_m acts on the sequence of Fourier coefficients $\varphi = \{\varphi_k\}$ as follows:

$$\Pi_m\varphi_k = (\Pi_m\varphi)_k = \begin{cases} \varphi_k & \text{if } |k| \le m \\ 0 & \text{if } |k| > m \end{cases}, \tag{4.2}$$

so that $\partial_t v_k^{(m)} \equiv 0$ for $|k| > m$. Consequently, system (4.1) is in principle a finite system of ODEs.

Theorem 4.1. *Let $s_0 \ge 0$ and $v(0) = v^0 \in \dot{H}^{s_0}$. Let $m \in \mathbb{N}$ be fixed. Then there exits $T^* > 0$, which might depend on m, such that system (4.1) has a unique solution on the time interval $[0, T^*]$. This solution $v = v^{(m)}(t)$ can be extended to the maximal interval $[0, T^*_{\max})$ such that either $T^*_{\max} = +\infty$ or $\limsup_{t\to T^*_{\max}-0} \|v^{(m)}(t)\|_{\dot{H}^0} = +\infty$.*

Proof. We observe that system (4.1) is essentially a finite system of ordinary differential equations with quadratic nonlinearity. Therefore one can guarantee short time existence and uniqueness of solution on an interval $[0, T^*]$ and the maximal interval of existence $[0, T^*_{\max})$. Observe, however, that in principle T^* and $T^*_{\max}$ may depend on m. But later we show that this is not the case. □

Next we show that $T^*_{\max} = +\infty$. To establish this and to be able to pass to the limit, as $m \to \infty$, we need to use global *a-priori* estimates for the solutions of (4.1).

Proposition 4.1. *Let $v^0 \in \dot{H}^0$. Then for every $m \in \mathbb{N}$ the solution $v = v^{(m)}(t)$ of (4.1) exists globally in time. Moreover, the $\dot{H}^0$-norm (that is, the L_2-norm) of v is constant in time:*

$$\|v^{(m)}(t)\|^2_{\dot{H}^0} = \|v(0)\|^2_{\dot{H}^0} = \|v^0\|^2_{\dot{H}^0} \quad \textit{for all} \ \ t \ge 0. \tag{4.3}$$

Proof. First we establish (4.3) for $t \in [0, T^*_{\max})$ (see Theorem 4.1). We observe that $\partial_t v_k^{(m)} = 0$, for all $|k| > m$, and hence $\|(I - \Pi_m)v^{(m)}(t)\|^2_{\dot{H}^0} = \|(I - \Pi_m)v^{(m)}(0)\|^2_{\dot{H}^0}$, for $t \in [0, T^*_{\max})$. Next we show that

$$\|\Pi_m v(t)\|^2_{\dot{H}^0} = \|\Pi_m v(0)\|^2_{\dot{H}^0}\ , \quad \text{for} \ \ t \in [0, T^*_{\max}). \tag{4.4}$$

For each $|k| \le m$, we multiply (4.1) by $\Pi_m\bar{v}_k = \Pi_m v_{-k}$ and the complex conjugate of equation (4.1) by $\Pi_m v_k$ and sum over all $|k| \le m$:

$$\partial_t\|\Pi_m v\|^2_{\dot{H}^0} = \sum_{|k|\le m} \partial_t v_k\bar{v}_k + v_k\partial_t\bar{v}_k =$$

$$\sum_{k\in\mathbb{Z}_0} (\Pi_m\bar{v}_k)\frac{ik}{2} \sum_{k_1+k_2=k,} e^{i3kk_1k_2t}\Pi_m v_{k_1}\Pi_m v_{k_2} - \sum_{k\in\mathbb{Z}_0} (\Pi_m v_k)\frac{ik}{2} \sum_{k_1+k_2=k,} e^{-i3kk_1k_2t}\Pi_m\bar{v}_{k_1}\Pi_m\bar{v}_{k_2}.$$

Elementary transformations yield

$$\partial_t\|\Pi_m v\|^2_{\dot H^0} = \sum_{k_1+k_2-k=0}\frac{ik}{2}e^{i3kk_1k_2t}\Pi_m v_{-k}\Pi_m v_{k_1}\Pi_m v_{k_2} - \sum_{k_1+k_2-k=0}\frac{ik}{2}e^{-i3kk_1k_2t}\Pi_m v_k\Pi_m v_{-k_1}\Pi_m v_{-k_2}.$$

Setting $k=-k_3$ we obtain

$$\partial_t\|\Pi_m v\|^2_{\dot H^0} = -\sum_{k_1+k_2+k_3=0}\frac{1}{2}ik_3e^{-i3k_3k_1k_2t}\Pi_m v_{k_3}\Pi_m v_{k_1}\Pi_m v_{k_2} + \sum_{k_1+k_2+k_3=0}\frac{1}{2}ik_3e^{i3k_3k_1k_2t}\Pi_m v_{-k_3}\Pi_m v_{-k_1}\Pi_m v_{-k_2},$$

and changing in the second sum k_j to $-k_j$ yields

$$\partial_t\|\Pi_m v\|^2_{\dot H^0} = -\sum_{k_1+k_2+k_3=0}ik_3e^{-i3k_3k_1k_2t}\Pi_m v_{k_3}\Pi_m v_{k_1}\Pi_m v_{k_2}. \tag{4.5}$$

Since $k_3=-k_1-k_2$, this equation is equivalent to

$$\partial_t\|\Pi_m v\|^2_{\dot H^0} = \sum_{k_1+k_2+k_3=0}ik_1e^{-i3k_3k_1k_2t}\Pi_m v_{k_3}\Pi_m v_{k_1}\Pi_m v_{k_2} + \sum_{k_1+k_2+k_3=0}ik_2e^{-i3k_3k_1k_2t}\Pi_m v_{k_3}\Pi_m v_{k_1}\Pi_m v_{k_2}.$$

Exchanging k_1 and k_3 in the first term and k_2 and k_3 in the second term we get

$$\partial_t\|\Pi_m v\|^2_{\dot H^0} = 2\sum_{k_1+k_2+k_3=0}ik_3e^{-i3k_3k_1k_2t}\Pi_m v_{k_3}\Pi_m v_{k_1}\Pi_m v_{k_2}. \tag{4.6}$$

Obviously, (4.5) and (4.6) imply that $\partial_t\|\Pi_m v\|^2_{\dot H^0}=0$, which gives (4.4) and, consequently,

$$\|v(t)\|^2_{\dot H^0} = \|v(0)\|^2_{\dot H^0} \quad \text{for} \quad t\in[0,T^*_{\max}).$$

Therefore, based on Theorem 4.1, $T^*_{\max}=\infty$, which completes the proof. □

Remark 4.1. We remark that a somewhat similar argument is used in [7] for the Fourier proof of the L_2-conservation property for the KdV equation on $\mathbb{R}$.

We now derive higher order Sobolev norm estimates for the solutions $v=v^{(m)}(t)$ of the truncated system (4.1). For this purpose we carry out the first and the second differentiation by parts for the truncated system (4.1), as we have done above in section 3.1. Similarly to (3.13) we obtain that solution $v^{(m)}(t)$, which was constructed in Theorem 4.1 and Proposition 4.1, satisfies the system

$$\partial_t\left(v^{(m)}_k - \frac{1}{6}B_{2,m}(v^{(m)},v^{(m)})_k\right) = \frac{i}{6}R_{3,m}(v^{(m)},v^{(m)},v^{(m)})_k, \quad k\in\mathbb{Z}_0, \tag{4.7}$$

where

$$B_{2,m}(v,v)_k = \Pi_m B_2(\Pi_m v,\Pi_m v)_k = \Pi_m\sum_{k_1+k_2=k}\frac{e^{i3kk_1k_2t}\Pi_m v_{k_1}\Pi_m v_{k_2}}{k_1k_2}, \tag{4.8}$$

and

$$R_{3,m}(v,v,v)_k = \Pi_m\sum_{k_1+k_2+k_3=k}\frac{e^{i3(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1}\Pi_m v_{k_1}\Pi_m v_{k_2}\Pi_m v_{k_3}. \tag{4.9}$$

After the second differentiation by parts we see that $v^{(m)}(t)$ satisfies the following finite dimensional analog of (3.31):

$$\partial_t \left(v_k^{(m)} - \frac{1}{6} B_{2,m}((v^{(m)})^2)_k - \frac{1}{18} B_{3,m}((v^{(m)})^3)_k \right) = \frac{i}{6} A_{\mathrm{res},m}(v^{(m)})_k + \frac{i}{18} B_{4,m}((v^{(m)})^4)_k, \tag{4.10}$$

where

$$B_{3,m}(v^3) = \Pi_m B_3((\Pi_m v)^3), \quad B_{4,m}(v^4) = \Pi_m B_4((\Pi_m v)^4). \tag{4.11}$$

Concerning the resonant operator, by Proposition 4.1 we have the conservation of energy for the truncated system (4.1); hence, arguing as in Remark 3.4 and similar to (3.22) we have

$$A_{\mathrm{res},m}(v)_k = \begin{cases} \frac{1}{k}(\Pi_m v_k)(\|v^0\|^2_{\dot H^0} - |\Pi_m v_k|^2) & |k| < m \\ 0 & |k| \geq m \end{cases}. \tag{4.12}$$

We now establish estimates for the solutions $v^{(m)}$ in higher order Sobolev spaces.

Theorem 4.2. *Let M_0 be given and let $s_0 \geq 0$. Assume that $v^{(m)}(0) = v^0 \in \dot H^{s_0}$ and suppose further that $\|v^{(m)}(0)\|_{\dot H^0} \leq M_0$. Let $T > 0$ and let $v^{(m)}(t)$ be a solution of (4.1) over $[0,T]$. Then $v^{(m)}(t)$ solves (4.10) and, uniformly in m, satisfies the estimate*

$$\|v^{(m)}(t)\|_{\dot H^{s_0}} \leq M_{s_0} \quad \text{for} \ \ 0 \leq t \leq T, \tag{4.13}$$

where M_{s_0} depends only on M_0, T, $\|v(0)\|_{\dot H^{s_0}}$ and s_0.

Proof. Since we are dealing with finite systems of ordinary differential equations and the sums are carried out over finite number of indices, it is clear that the solution of (4.1) solves (4.10). Moreover, thanks to Proposition 4.1 we have

$$\|v^{(m)}(t)\|_{\dot H^0} = \|v^{(m)}(0)\|_{\dot H^0} \leq M_0. \tag{4.14}$$

Since $\partial_t v_k^{(m)} = 0$ for $|k| > m$, it follows that

$$\|(I - \Pi_m)v^{(m)}(t)\|_{\dot H^{s_0}} = \|(I - \Pi_m)v^{(m)}(0)\|_{\dot H^{s_0}} \leq \|v^0\|_{\dot H^{s_0}}. \tag{4.15}$$

Therefore, we need to focus on estimating $\|\Pi_m v^{(m)}\|_{\dot H^{s_0}}$. We set in (4.10)

$$z_k^{(m)} = v_k^{(m)} - w_k^{(m)}(v^{(m)}) - \varphi_k^{(m)}(v^{(m)}), \ k \in \mathbb{Z}_0,$$

where

$$w^{(m)}(v^{(m)}) = \frac{1}{6} B_{2,m}(v^{(m)}, v^{(m)}), \quad \varphi^{(m)}(v^{(m)}) = \frac{1}{18} B_{3,m}(v^{(m)}, v^{(m)}, v^{(m)})$$

and

$$Q_m(v^{(m)})_k = \frac{i}{6} A_{\mathrm{res,m}}(v^{(m)})_k + \frac{i}{18} B_{4,m}(v^{(m)}, v^{(m)}, v^{(m)}, v^{(m)})_k, \quad k \in \mathbb{Z}_0.$$

Then (4.10) goes over to

$$\partial_t z_k^{(m)} = Q_m(v^{(m)})_k = Q_m(z^{(m)} + w^{(m)}(v^{(m)}) + \varphi^{(m)}(v^{(m)}))_k \tag{4.16}$$

with $z^{(m)}(0) = z^0 = v^0 - w^{(m)}(v^0) - \varphi^{(m)}(v^0)$, where $z^{(m)}(0) \in \dot H^{s_0}$ in view of Lemmas 7.2 and 7.6. We observe that the operators $B_{2,m}$, $B_{3,m}$ and $B_{4,m}$ in (4.10) satisfy the same estimates in the same spaces as B_2, B_3 and B_4, respectively, and the estimates are uniform in m. We fix a positive integer n_0 so that $s_0/n_0 = \varepsilon_0 < 1/2$. Our goal is to establish an explicit bound for $\sup_{0 \leq t \leq T} \|\Pi_m v^{(m)}(t)\|_{\dot H^{s_0}}$. This is done in n_0 steps. By Lemmas 7.11 and 7.14 below, Q_m is bounded from $\dot H^s$ to $\dot H^{s+\varepsilon}$, for $s \geq 0$ and $0 < \varepsilon < 1/2$:

$$\|Q_m(v^{(m)})\|_{\dot H^{s+\varepsilon}} \leq \frac{1}{18} c_4(s,\varepsilon) \|v^{(m)}\|^4_{\dot H^s} + \frac{1}{6} c_5(s) \|v^{(m)}\|_{\dot H^s}. \tag{4.17}$$

In view of (4.14) and (4.17) the right-hand side in (4.16) is bounded in $\dot H^{\varepsilon_0}$ (we take $s = 0$ and $\varepsilon = \varepsilon_0$ in (4.17)):

$$\|Q_m(v^{(m)}(t))\|_{\dot H^{\varepsilon_0}} \le C_{\varepsilon_0} = C_{\varepsilon_0}(M_0).$$

We take the scalar product of (4.16) and $\Pi_m z(t)$ in $\dot H^{\varepsilon_0}$, that is, we multiply the equation (4.16) by $|k|^{2\varepsilon_0}\bar z_k$ and sum the results over all $|k| \le m$. We obtain

$$\partial_t \|\Pi_m z^{(m)}\|^2_{\dot H^{\varepsilon_0}} = 2(Q_m(v^{(m)}), \Pi_m z^{(m)})_{\dot H^{\varepsilon_0}} \le \|Q_m(v^{(m)})\|^2_{\dot H^{\varepsilon_0}} + \|\Pi_m z^{(m)}\|^2_{\dot H^{\varepsilon_0}},$$

where $\Pi_m z(0)$ is bounded in $\dot H^{s_0} \subseteq \dot H^{\varepsilon_0}$ since $s_0 \ge \varepsilon_0$. By the Gronwall inequality

$$\|\Pi_m z^{(m)}(t)\|_{\dot H^{\varepsilon_0}} \le C_{\varepsilon_0}(T, M_0), \quad t \in [0, T],$$

and therefore $\Pi_m v^{(m)}(t) = \Pi_m z^{(m)}(t) + w^{(m)}(v^{(m)}(t)) + \varphi^{(m)}(v^{(m)}(t))$ is bounded in $\dot H^{\varepsilon_0}$ for $t \in [0, T]$ with bound that depends only on T, M_0 and $\|v^0\|_{\dot H^{s_0}}$ (note that $w^{(m)}(v^{(m)})$ and $\varphi^{(m)}(v^{(m)})$ for $v^{(m)} \in L_2$ are bounded in $\dot H^1$ and $\dot H^2$, respectively, see Lemmas 7.2 and 7.6 below; and thus they are bounded in $\dot H^{\varepsilon_0}$ since $\varepsilon_0 < 1/2$). Consequently, $Q_m(v^{(m)}(t))$ on the right-hand side in (4.16) is bounded in $\dot H^{2\varepsilon_0}$ (thanks to (4.17)) and taking the scalar product of (4.16) and $\Pi_m z^{(m)}$ in $\dot H^{2\varepsilon_0}$ we obtain, as above, that $\Pi_m z^{(m)}(t)$ and, hence, $\Pi_m v^{(m)}(t)$ are bounded in $H^{2\varepsilon_0}$ uniformly for $t \in [0, T]$ with bound that depends only on T, M_0 and $\|v^0\|_{\dot H^{s_0}}$. After n_0 steps we get on the interval $[0, T]$

$$\|\Pi_m v^{(m)}(t)\|_{\dot H^{s_0}} \le C_{s_0} = C_{s_0}(T, \|v^0\|_{\dot H^{s_0}}, M_0), \tag{4.18}$$

which together with (4.15) completes the proof of the theorem. □

Proposition 4.2. *Let $T > 0$, $s_0 \ge 0$ and $S > 3/2$. Then the solutions $v = v^{(m)}$ of (4.1) with $v(0) \in \dot H^{s_0}$ satisfy, uniformly in m, T, and s_0, the estimate*

$$\|\partial_t v^{(m)}(t)\|_{\dot H^{-S}} \le c_1(S-1)\|v(0)\|^2_{\dot H^0}. \tag{4.19}$$

Proof. This immediately follows from Lemma 7.1 and the conservation of energy of the solutions $v^{(m)}$ of (4.1), see Proposition 4.1. The constant $c_1(S-1)$ is also defined in Lemma 7.1. □

We are now ready to establish the existence of weak solutions to equation (3.8) as the limit of a subsequence of $v^{(m)}(t)$, the solutions of the truncated system (4.1). We first deal with the case $s > 0$. The case $s = 0$ is considered in Theorem 6.4 below.

Theorem 4.3. *Let $s_0 > 0$, $v(0) \in \dot H^{s_0}$ and let $T > 0$ be fixed. Let σ be fixed so that $0 < \sigma < s_0$. Then there exists a subsequence of $v^{(m)}(t)$ (which we still label by (m)) of solutions of (4.1), with the same initial data $v(0) \in \dot H^{s_0}$, such that $v^{(m)}(t)$ converges strongly in $L_p([0,T]; H^\sigma)$, for any fixed $1 < p < \infty$, and $*$-weakly in $L_\infty([0,T]; \dot H^{s_0})$ to $v^\infty(t)$; and $v^\infty(t)$ is a solution of (3.8) in the sense of Definition 3.1 and is bounded in $L_\infty([0,T], \dot H^{s_0})$ with norm*

$$\|v^\infty\|_{L_\infty([0,T],\dot H^{s_0})} \le M_{s_0}, \tag{4.20}$$

where M_{s_0} is as in (4.13). Moreover,

$$\|v^\infty(t)\|_{\dot H^0} = \|v^0\|_{\dot H^0} \quad a.e.\ in \quad [0, T]. \tag{4.21}$$

Proof. We first observe that based on Proposition 4.1 the solution $v^{(m)}(t)$ of (4.1) exists globally in time. Next we set $\theta = -S$, where $S > 3/2$ is as in Proposition 4.2. It follows from (4.13) and (4.19) that $v^{(m)}$ is bounded in $L_\infty([0,T]; H^{s_0}) \subset L_p([0,T]; H^{s_0})$; while the time derivative $\partial_t v^{(m)}$ is bounded in $L_\infty([0,T]; H^\theta) \subset L_p([0,T]; H^\theta)$ uniformly with respect

to m. Since the imbedding $\dot{H}^\sigma \subset \dot{H}^\theta$ is compact, the existence of a subsequence that converges strongly in $L_p([0,T];H^\sigma)$ and $*$-weakly in $L_\infty([0,T];\dot{H}^{s_0})$ to a function v^∞ follows from the classical compactness theorem (Aubin compactness theorem). Furthermore, v^∞ is bounded and continuous with values in $\dot{H}^\theta$ and is bounded and weakly continuous with values in $\dot{H}^\gamma$ for any γ, $\theta \le \gamma \le s_0$ (see, for instance, [10], [22]).

Since $v^{(m)}$ converges to v^∞ strongly in $L_p([0,T],\dot{H}^\sigma)$ with $\sigma > 0$, it follows that there is a subsequence, also denoted by $v^{(m)}(t)$, which converges to $v^\infty(t)$ strongly in $\dot{H}^\sigma$ and hence strongly in $\dot{H}^0$ for almost every $t \in [0,T]$. However, $\|v^{(m)}(t)\|_{\dot{H}^0} = \|v(0)\|_{\dot{H}^0}$ for every t. Therefore $\|v^\infty(t)\|_{\dot{H}^0} = \|v(0)\|_{\dot{H}^0}$ for almost every $t \in [0,T]$, which proves (4.21).

Every $v_k^{(m)}(t)$ is a solution of (4.1), which we write, using (3.11), as follows

$$v_k^{(m)}(t) - v_k(0) = \int_0^t \Pi_m B_1(\Pi_m v^{(m)}(\tau), \Pi_m v^{(m)}(\tau))_k d\tau. \tag{4.22}$$

Therefore, using the symmetry of B_1 and setting $\Pi_{-m} = I - \Pi_m$ we obtain

$$\begin{aligned} v_k^{(m)}(t) - v_k^\infty(t) + v_k^\infty(t) - v_k(0) = \\ \int_0^t \Pi_m B_1\big(\Pi_m(v^{(m)}(\tau) - v^\infty(\tau)), \Pi_m(v^{(m)}(\tau) + v^\infty(\tau))\big)_k d\tau - \\ \int_0^t \Pi_m B_1\big(\Pi_{-m} v^\infty(\tau), \Pi_m v^\infty(\tau) + v^\infty(\tau)\big)_k d\tau - \\ \int_0^t \Pi_{-m} B_1\big(v^\infty(\tau), v^\infty(\tau)\big)_k d\tau + \\ \int_0^t B_1\big(v^\infty(\tau), v^\infty(\tau)\big)_k d\tau. \end{aligned} \tag{4.23}$$

By Lemma 7.1 and the readily established convergence of $v^{(m)} \to v^\infty$ in $L_p([0,T];\dot{H}^\sigma)$ and the fact that $v^\infty \in L_\infty([0,T];\dot{H}^{s_0})$ we observe that the first three terms on the right-hand side converge to zero. For example, in view of Lemma 7.1 the first term on the right-hand side is less in absolute value than

$$\begin{aligned} |k|^{-\theta} \int_0^t \|B_1\big(\Pi_m(v^{(m)}(\tau) - v^\infty(\tau)), \Pi_m(v^{(m)}(\tau) + v^\infty(\tau))\big)\|_{\dot{H}^\theta} d\tau \le \\ c_1(\theta)|k|^{-\theta} \int_0^t \|v^{(m)}(\tau) - v^\infty(\tau)\|_{\dot{H}^0} \|v^{(m)}(\tau) + v^\infty(\tau)\|_{\dot{H}^0} d\tau \le \\ 2c_1(\theta)|k|^{-\theta} \|v(0)\|_{\dot{H}^0} \|v^{(m)} - v^\infty\|_{L_1([0,T],\dot{H}^0)} \to 0 \quad \text{as} \quad m \to \infty. \end{aligned}$$

The second term is treated similarly. By the definition of Π_{-m} the third term is identically zero for $m \ge |k|$. Since $v_k^{(m)}(t) \to v_k^\infty(t)$, as $m \to \infty$, for each $k \in \mathbb{Z}_0$ and almost every $t \in [0,T]$, passing to the limit in (4.23) we obtain the equality

$$v_k^\infty(t) - v_k(0) = \int_0^t B_1(v^\infty(\tau), v^\infty(\tau))_k d\tau, \tag{4.24}$$

which holds for almost every t. As in Remark 3.1 we see that (4.24) holds for every t and v^∞ is a solution of (3.9) in the sense of Definition 3.1. □

Corollary 4.1. *Let $s_0 > 0$, $v(0) \in \dot{H}^{s_0}$. The solution $v^\infty \in L_\infty([0,T];\dot{H}^{s_0})$ constructed in Theorem 4.3 also satisfies the integrated first form of the KdV (3.18) and the integrated second form of the KdV equation (3.32).*

Proof. Applying the formal steps introduced in section 3 to the solutions $v^{(m)}(t)$ of equation (4.1) one can rigorously show that $v^{(m)}$ is the solution of the corresponding Galerkin versions of (3.18) and (3.32). Therefore we can pass to the limit along a subsequence, as $m \to \infty$, in the same way as we did in (4.23). We also observe that by the continuity properties of the operators B_2 and R_3 in (3.18) and B_2, B_3, B_4, and A_{res} in (3.32), respectively, all the terms in (3.18) are bounded from $\dot{H}^s$ to $\dot{H}^s$, for $s > 1/2$, while the terms in (3.32) are bounded from $\dot{H}^s$ to $\dot{H}^s$ for $s \geq 0$. $\square$

5. Global well-posedness and Lipschitz continuity (regular case, $s > 1/2$)

Here we will consider regular initial data in $\dot{H}^s$, for $s > 1/2$, and will work with equation (3.18). Less regular data will be considered in section 6.

By Corollary 4.1 there exists a solution of (3.18). For the proof of the uniqueness we need the following auxiliary result. We introduce the projection Π_0 on the zero Fourier mode, $\Pi_0(\{g_k, k \in \mathbb{Z}\}) = g_0$.

Lemma 5.1. *Let $\varphi \in \dot{H}^{s-1}$, $s > 1/2$. Then the linear operator $L_\varphi : \dot{H}^\theta \to \dot{H}^\theta$, for $\theta > -1/2$,*

$$L_\varphi v = v - B_2(\varphi, v), \qquad (L_\varphi v)_k = v_k - B_2(\varphi, v)_k \quad k \in \mathbb{Z}_0, \tag{5.1}$$

where $B_2(u,v)_k = B_2(u,v,t)_k$ is defined in (3.16), has range $\dot{H}^\theta$ for every fixed t. Moreover, for every $f \in \dot{H}^\theta$ the equation

$$L_\varphi v = f$$

has a unique solution $v = L_\varphi^{-1} f \in \dot{H}^\theta$, and the following estimate holds

$$\|v\|_{\dot{H}^\theta} \leq \|L_\varphi^{-1}\|_{\mathcal{L}(\dot{H}^\theta)} \|f\|_{\dot{H}^\theta}. \tag{5.2}$$

Furthermore, for every $\varphi \in \dot{H}^{-1} \cap \dot{H}^{s-2}$, with $s > 1/2$, the inverse operator L_φ^{-1} can be extended as a bounded linear map from $\dot{H}^\theta$ into itself, for $\theta \in [-1, s]$, with norm satisfying the estimate

$$\|L_\varphi^{-1}\|_{\mathcal{L}(\dot{H}^\theta)} \leq F(\|\varphi\|_{\dot{H}^{-1}}, \|\varphi\|_{\dot{H}^{s-2}}), \tag{5.3}$$

where F is a monotonic increasing function in each argument, which is independent of t.

Remark 5.1. Note that if $s \leq 1$, then clearly $F(\|\varphi\|_{\dot{H}^{-1}}, \|\varphi\|_{\dot{H}^{s-2}}) \leq F(\|\varphi\|_{\dot{H}^{-1}}, \|\varphi\|_{\dot{H}^{-1}})$.

Proof. We first show that the homogeneous equation

$$L_\varphi v = v - B_2(\varphi, v) = 0, \quad v \in \dot{H}^\theta \ \text{ for } \ \theta > -1/2 \tag{5.4}$$

has only the trivial solution $v = 0$. Note that according to Lemma 7.2 the bilinear operator $B_2(\varphi, v)$ is bounded from $\dot{H}^\theta \times \dot{H}^\theta$ into $\dot{H}^{\theta+1}$, for $\theta > -1/2$. Therefore, if $v \in \dot{H}^\theta$ is a solution of (5.4), then $v \in \dot{H}^{\min(\theta, s-1)+1}$. Hence, it is sufficient to consider solutions v in $\dot{H}^\sigma$, for $\sigma > 1/2$.

In terms of the Fourier coefficients (5.4) reads

$$v_k - \sum_{k_1+k_2=k} \frac{e^{i3kk_1k_2t} \varphi_{k_1} v_{k_2}}{k_1 k_2} = 0, \quad k \in \mathbb{Z}_0.$$

Setting $u_k = e^{-ik^3t} v_k$ and $\psi_k = e^{-ik^3t} \varphi_k$ and taking into account (3.7) we obtain

$$u_k - \sum_{k_1+k_2=k} \frac{\psi_{k_1} u_{k_2}}{k_1 k_2} = 0 \tag{5.5}$$

with $\|u\|_{\dot{H}^s} = \|v\|_{\dot{H}^s}$. We now define $w(x)$ and $\xi(x)$ by setting

$$w_k = \frac{u_k}{ik}, \quad \xi_k = \frac{\psi_k}{ik}, \; k \in \mathbb{Z}_0, \quad w_0 = \xi_0 = 0.$$

By definition, both w and ξ have mean value zero: $\int_0^{2\pi} w(x)dx = \int_0^{2\pi} \xi(x)dx = 0$ and, in addition, $w'(x) = u(x)$, $\xi'(x) = \psi(x)$ so that $\|u\|_{\dot{H}^s} = \|w\|_{\dot{H}^{s+1}}$, $\|\psi\|_{\dot{H}^s} = \|\xi\|_{\dot{H}^{s+1}}$. Multiplying (5.5) by e^{ikx} and summing with respect to *all* $k \in \mathbb{Z}$ we see that

$$w(x) = \sum_{k \in \mathbb{Z}_0} \frac{u_k}{ik} e^{ikx}, \qquad w'(x) = u(x),$$

satisfies the following boundary value problem:

$$\begin{gathered} w'(x) + \xi(x)w(x) = \frac{1}{2\pi}\int_0^{2\pi} \xi(x)w(x)dx, \\ w \text{ is periodic, with period } 2\pi, \quad \text{and} \quad \int_0^{2\pi} w(x)dx = 0, \end{gathered} \tag{5.6}$$

in the classical sense since it is given that $\xi \in H^{s+1} \subset C^1$, for $s > 1/2$, and we are looking for a solution $w \in H^{\theta+1} \subset C^1$, for $\theta > 1/2$. Note that the right-hand side of (5.6) is the projection $\Pi_0(\xi w)$. Then

$$w'(x) + \xi(x)w(x) = \tilde{c},$$

where

$$\tilde{c} = \Pi_0(\xi w) = \frac{1}{2\pi}\int_0^{2\pi} \xi(x)w(x)dx \tag{5.7}$$

and we can treat this equation as a first-order linear ordinary differential equation with general solution

$$w(x) = Ce^{-\int_0^x \xi(\sigma)d\sigma} + \tilde{c}\int_0^x e^{\int_x^r \xi(\sigma)d\sigma} dr, \quad C \in \mathbb{R}.$$

Since only the first term on the right-hand side is periodic, it follows that $w(x)$ is periodic if and only if $\tilde{c} = 0$, and then in this case $w(x)$ has mean value zero if and only if $C = 0$. Hence, $w = 0$ and, therefore, $u = 0$ and $v = 0$.

Now we prove that the range $L_\varphi \dot{H}^\theta$ of L_φ coincides with $\dot{H}^\theta$, and that L_φ has a bounded inverse. For $s > \theta$ this follows from the Fredholm theory. Observe that the embedding $\dot{H}^\sigma \to \dot{H}^{\sigma-\varepsilon}$ is compact if $\varepsilon > 0$. By Lemma 7.2, $B_2(\varphi, v)$ is bounded from $\dot{H}^{s-1} \times \dot{H}^\theta$ into $\dot{H}^{\theta+1}$ for $\theta \le s-1$, while for $\theta > s-1$ it is bounded from $\dot{H}^{s-1} \times \dot{H}^\theta$ into $\dot{H}^s$. Hence $B_2(\varphi, \cdot)$ is compact from $\dot{H}^\theta$ into itself and $L_\varphi = I - B_2(\varphi, \cdot)$ is a Fredholm operator in $\dot{H}^\theta$, the index (dimension of nullspace minus codimension of range) of this operator is zero. Therefore, the range $L_\varphi \dot{H}^\theta$ is a closed subspace in $\dot{H}^\theta$ with finite codimension, and the codimension equals dimension of the nullspace of L_φ. We already have proven that the nullspace is trivial, therefore the codimension is zero and $L_\varphi \dot{H}^\theta = \dot{H}^\theta$. Hence, the inverse operator L_φ^{-1} exists and is bounded.

Note that L_φ depends on t as well as on φ and we have boundedness of L_φ^{-1} for every given t and φ. Next we prove a uniform estimate for L_φ^{-1} for $t \in [0, T]$ and φ satisfying $\|\varphi\|_{\dot{H}^{s-1}} \le C$. To estimate the norm of L_φ^{-1} is sufficient to consider it on a dense set $\dot{H}^s$, $s > 1/2$, of smooth enough functions v. For such functions we write inverse operator L_φ^{-1} explicitly and obtain the explicit norm estimate (5.3). To prove (5.2) and (5.3) we consider for $\varphi \in \dot{H}^{s-1}$ and $f \in \dot{H}^s$ the nonhomogeneous equation

$$L_\varphi v = v - B_2(\varphi, v) = f.$$

For $k \in \mathbb{Z}_0$ we set $u_k = e^{-ik^3t}v_k$, $\psi_k = e^{-ik^3t}\varphi_k$, $g_k = e^{-ik^3t}f_k$ and write the equation for w:

$$w' + \xi w = g + \tilde{c},$$

where, as before, $w' = u$, $\xi' = \psi$ and w, ξ, g depend on t as a parameter. Since g has mean value zero, the constant $\tilde{c} = \tilde{c}(w)$ as before satisfies (5.7). The general solution is

$$w(x) = \left(\int_0^x (g(r) + \tilde{c})e^{\Xi(r)}dr + C \right) e^{-\Xi(x)}, \quad \Xi(x) = \int_0^x \xi(\sigma)d\sigma, \tag{5.8}$$

where $C \in \mathbb{R}$ is a constant. Since ξ is periodic with mean value zero, $\Xi(x)$ is also periodic. The condition on w to be periodic gives

$$\int_0^{2\pi} (g(r) + \tilde{c})e^{\Xi(r)}dr = 0,$$

which uniquely defines $\tilde{c}$ in terms of g and ξ:

$$\tilde{c} = -\frac{\int_0^{2\pi} g(r)e^{\Xi(r)}dr}{\int_0^{2\pi} e^{\Xi(r)}dr}. \tag{5.9}$$

We observe that w defined in (5.8) satisfies (5.7) (for any C). The condition $\int_0^{2\pi} w(x)dx = 0$ uniquely defines the constant C:

$$C = -\frac{\int_0^{2\pi} e^{-\Xi(x)} \int_0^x (g(r) + \tilde{c})e^{\Xi(r)}drdx}{\int_0^{2\pi} e^{-\Xi(x)}dx}, \tag{5.10}$$

and, hence, the solution $w(x)$ of (5.8) is uniquely defined. We denote

$$G(r) = \int_0^r g(r_1)dr_1,$$

since g is periodic with mean value zero, $G(r)$ is also periodic, with $G(0) = G(2\pi)$. Integrating by parts we get

$$\int_0^x g(r)e^{\Xi(r)}dr = \int_0^x e^{\Xi(r)}dG(r) = G(x)e^{\Xi(x)} - \int_0^x G(r)\xi(r)e^{\Xi(r)}dr \tag{5.11}$$

and rewrite (5.8) in the form

$$w(x) = G(x) + \left(C - \int_0^x G(r)\xi(r)e^{\Xi(r)}dr + \tilde{c}\int_0^x e^{\Xi(r)}dr \right) e^{-\Xi(x)}. \tag{5.12}$$

Using (5.11) and $G(2\pi) = 0$ we write (5.9) and (5.10) as follows:

$$\tilde{c} = \frac{\int_0^{2\pi} G(r)\xi(r)e^{\Xi(r)}dr}{\int_0^{2\pi} e^{\Xi(r)}dr},$$

$$C = \frac{\int_0^{2\pi} e^{-\Xi(x)} \int_0^x G(r)\xi(r)e^{\Xi(r)}drdx - \tilde{c}\int_0^{2\pi} e^{-\Xi(x)} \int_0^x e^{\Xi(r)}drdx - \int_0^{2\pi} G(x)dx}{\int_0^{2\pi} e^{-\Xi(x)}dx}.$$

According to the Sobolev embedding theorem, since $s > 1/2$

$$-c(s)\|\xi\|_{\dot{H}^{s-1}} \le \Xi(r) \le c(s)\|\xi\|_{\dot{H}^{s-1}}, \quad \|\xi\|_{\dot{H}^{s-1}} = \|\varphi\|_{\dot{H}^{s-2}}.$$

Therefore, $e^{|\Xi(x)|} \le e^{c(s)\|\varphi\|_{\dot{H}^{s-2}}}$, $e^{-|\Xi(x)|} \ge e^{-c(s)\|\varphi\|_{\dot{H}^{s-2}}}$ and the denominators in (5.9) and (5.10) are bounded away from zero and we obtain by the Cauchy–Schwarz inequality

(recalling that the norms are normalized)

$$|\tilde{c}| \leq \frac{1}{2\pi} e^{2c(s)\|\varphi\|_{\dot{H}^{s-2}}} \int_0^{2\pi} G(r)\xi(r)dr \leq$$
$$e^{2c(s)\|\varphi\|_{\dot{H}^{s-2}}} \|G\|_{\dot{H}^0} \|\xi\|_{\dot{H}^0} = e^{2c(s)\|\varphi\|_{\dot{H}^{s-2}}} \|g\|_{\dot{H}^{-1}} \|\varphi\|_{\dot{H}^{-1}},$$

and using this we get the following estimate for C

$$|C| \leq \frac{1}{2\pi} e^{c(s)\|\varphi\|_{\dot{H}^{s-2}}} \bigg(\int_0^{2\pi} e^{-\Xi(x)} \int_0^x |G(r)\xi(r)| \, e^{\Xi(r)} dr dx +$$
$$|\tilde{c}| \int_0^{2\pi} e^{-\Xi(x)} \int_0^x e^{\Xi(r)} dr dx + 2\pi \|G\|_{\dot{H}^0} \bigg) \leq$$
$$e^{c(s)\|\varphi\|_{\dot{H}^{s-2}}} \left(e^{2c(s)\|\varphi\|_{\dot{H}^{s-2}}} \|G\|_{\dot{H}^0} \|\xi\|_{\dot{H}^0} + |\tilde{c}| 2\pi e^{2c(s)\|\varphi\|_{\dot{H}^{s-2}}} + \|G\|_{\dot{H}^0} \right) \leq$$
$$\leq 2(\pi+1) e^{5c(s)\|\varphi\|_{\dot{H}^{s-2}}} \|g\|_{\dot{H}^{-1}} (\|\varphi\|_{\dot{H}^{-1}} + 1).$$

To estimate the operator norm we use the well-known Banach algebra property (7.5) of the Sobolev spaces H^s for $s > 1/2$ (see also Corollary 7.1) and expanding the exponential into Taylor series we have

$$\left\| e^{\pm\Xi(x)} \right\|_{H^s} \leq K_1(s)^{-1} \big(e^{K_1(s)\|\xi\|_{\dot{H}^{s-1}}} + K_1(s) - 1 \big) = K_1(s)^{-1} \left(e^{K_1(s)\|\varphi\|_{\dot{H}^{s-2}}} + K_1(s) - 1 \right).$$

Therefore, from (5.12) we get for $0 \leq s_0 \leq s$

$$\|w\|_{\dot{H}^{s_0}} \leq \|G\|_{\dot{H}^{s_0}} + \max_{0 \leq x \leq 2\pi} \left| C - \int_0^x G(r)\xi(r) e^{\Xi(r)} dr + \tilde{c} \int_0^x e^{\Xi(r)} dr \right| \|e^{-\Xi(x)}\|_{\dot{H}^s},$$

and from the previous estimates we infer that

$$\|v\|_{\dot{H}^{s_0-1}} = \|w\|_{\dot{H}^{s_0}} \leq$$
$$\|g\|_{\dot{H}^{s_0-1}} + (6\pi+2) e^{5c(s)\|\varphi\|_{\dot{H}^{s-2}}} \|g\|_{\dot{H}^{-1}} (\|\varphi\|_{\dot{H}^{-1}} + 1) K_1(s)^{-1} \left(e^{K_1(s)\|\varphi\|_{\dot{H}^{s-2}}} + K_1(s) - 1 \right),$$

or in a more concise form

$$\|v\|_{\dot{H}^{s_0-1}} = \|w\|_{\dot{H}^{s_0}} \leq \|g\|_{\dot{H}^{s_0-1}} + \|g\|_{\dot{H}^{-1}} F_1 \left(\|\varphi\|_{\dot{H}^{-1}}, \|\varphi\|_{\dot{H}^{s-2}} \right), \tag{5.13}$$

for some (explicitly known) function F_1. Since $\|g\|_{\dot{H}^{-1}} \leq \|g\|_{\dot{H}^{s_0-1}}$ and $\|f\|_{\dot{H}^\theta} = \|g\|_{\dot{H}^\theta}$, we obtain (5.3) with $\theta = s_0 - 1$. The proof is complete. □

Remark 5.2. The estimate (5.3) for negative values of θ will be used in Theorem 6.5. Otherwise, for $\theta \geq 0$ a simpler form of this estimate, i.e., $\|L_\varphi^{-1}\|_{\mathcal{L}(\dot{H}^\theta)} \leq F(\|\varphi\|_{\dot{H}^0})$, satisfies all our needs.

Remark 5.3. In terms of w this lemma establishes the invertibility of the linearization of the Miura transform $M[w] = w' + w^2$. See [9] for the results on the invertibility of the Miura transform itself.

We now prove global well-posedness in $\dot{H}^s$ for $s > 1/2$.

Theorem 5.1. *Let $s > 1/2$, $v(0) = v^0 \in \dot{H}^s$, and let $T > 0$ be fixed. Then the solution $v = v^\infty$ of (3.9), in the sense of Definition 3.1, which was constructed in Theorem 4.3 is unique. Moreover, the solution is of the class $C([0,T];\dot{H}^s)$, and depends Lipschitz continuously on the initial data in the sense that is described in (5.18) below. Furthermore, $\|v^\infty(t)\|_{\dot{H}^0} = \|v(0)\|_{\dot{H}^0}$ for all $t \in [0,T]$.*

Proof. In view of Corollary 4.1 the solution $v = v^\infty$ of equation (3.9), which was constructed in Theorem 4.3, also satisfies equation (3.18) for all $t \in [0,T]$. Setting

$$y(t) = v(t) - v^0, \qquad y(0) = 0, \tag{5.14}$$

then by Theorem 4.3 $y \in L_\infty([0,T]; \dot{H}^s)$. Using the symmetry of B_2, $B_2(u,v) = B_2(v,u)$, we have from (3.18)

$$y(t) - \frac{1}{3}B_2(v^0, y(t)) = \frac{1}{6}B_2(y(t),y(t)) + \frac{i}{6}\int_0^t R_3(\left(y(\tau)+v^0\right)^3)d\tau. \tag{5.15}$$

Setting $L_{v^0}\, y = y - \frac{1}{3}B_2(v^0, y)$, then by virtue of Lemma 5.1, Lemma 7.2 and Lemma 7.15 we have

$$y(t) = L_{v^0}^{-1}(t)\left(\frac{1}{6}B_2(y(t),y(t)) + \frac{i}{6}\int_0^t R_3(\left(y(\tau)+v^0\right)^3)d\tau\right) =: \mathcal{F}(y)(t). \tag{5.16}$$

Let $T^* \in (0,T]$, to be determined later. We consider the Banach space $C([0,T^*]; \dot{H}^s)$ and the subspace $C_0([0,T^*]; \dot{H}^s) = \{y \in C([0,T^*]; \dot{H}^s) : \; y(0) = 0\}$. Next, we show that the nonlinear operator $\mathcal{F}$ maps the ball of radius A, that is, $\{y \in C_0([0,T^*]; \dot{H}^s) : \|y\|_{C([0,T^*];\dot{H}^s)} \le A\}$, into itself and is a contraction map provided that A and T^* are small enough. In fact, for A and T^* small enough we obtain by Lemma 5.1, Lemma 7.2 and Lemma 7.15 that

$$\begin{aligned}
\|\mathcal{F}(y)(t)\|_{\dot{H}^s} &\le c_7(s)\|L_{v^0}^{-1}\|_{\mathcal{L}(\dot{H}^s)}\left(A^2 + T^*(A + \|v^0\|_{\dot{H}^s})^3\right) \le \frac{A}{4},\\
\|\mathcal{F}(y_1)(t) - \mathcal{F}(y_2)(t)\|_{\dot{H}^s} &\le c_7(s)\|L_{v^0}^{-1}\|_{\mathcal{L}(\dot{H}^s)}\left(A + T^*(A + \|v^0\|_{\dot{H}^s})^2\right)\|y_1 - y_2\|_{C([0,T^*];\dot{H}^s)} \le\\
&\le \frac{1}{2}\|y_1 - y_2\|_{C([0,T^*];\dot{H}^s)},
\end{aligned} \tag{5.17}$$

where $c_7(s) = (5/3)\max(c_2(s), c_6(s))$. We now fix A and T^* small enough such that the above inequality holds. (Note that A depends only on $\|v^0\|_{\dot{H}^s}$ and therefore T^* also depends only on $\|v^0\|_{\dot{H}^s}$.) Hence by the Banach Contraction Principle there exists a unique solution $y(t)$ of (5.16) on the interval $[0,T^*]$.

Denote by $[0,T^*_{\max})$ the maximal interval of existence of the solutions of (5.16). By short time existence so obtained we have $T^*_{\max} > 0$. If $T^*_{\max} > T$ we are done with the proof. However, if $T^*_{\max} \le T$ then by standard arguments one can show that the $\limsup_{t\to T^*_{\max}-0}\|v(t)\|_{\dot{H}^s} = \infty$. Based on Theorem 4.3 and the uniqueness of solutions established above we have $v^\infty(t) = v(t) = y(t) + v_0$ for all $t \in [0, T^*_{\max})$. By virtue of (4.20) $\limsup_{t\to T^*_{\max}-0}\|v(t)\|_{\dot{H}^s} < \infty$. Consequently, $T^*_{\max} > T$. Hence $v = v^\infty \in C([0,T], \dot{H}^s)$ is unique on $[0,T]$. In particular, thanks to (4.21) and the above we have $\|v^\infty(t)\|_{\dot{H}^0} = \|v(0)\|_{\dot{H}^0}$, for all $t \in [0,T]$, i.e., $v^\infty(t)$ conserves the energy for all $t \in [0,T]$.

To prove the continuous dependence on the initial data (in fact, Lipschitz continuous) we consider two solutions $w(t)$ and $v(t)$ evolving from two close initial points $w(0) = w^0$ and $v(0) = v^0$. Then $z(t) = w(t) - w^0$ and $y(t) = v(t) - v^0$ satisfy on $[0,T^*]$, where T^* to be chosen later, the equations

$$\begin{aligned}
z(t) &= \frac{1}{6}B_2(z(t),z(t)) + \frac{1}{3}B_2(w^0, z(t)) + \frac{i}{6}\int_0^t R_3((z(\tau)+w^0)^3)d\tau,\\
y(t) &= \frac{1}{6}B_2(y(t),y(t)) + \frac{1}{3}B_2(v^0, y(t)) + \frac{i}{6}\int_0^t R_3((y(\tau)+v^0)^3)d\tau.
\end{aligned}$$

Therefore $\varphi(t) = z(t) - y(t)$ satisfies

$$\varphi(t) - \frac{1}{3}B_2(w^0, \varphi(t)) = \frac{1}{6}B_2(z(t) + y(t), \varphi(t)) + \frac{1}{3}B_2(w^0 - v^0, y(t)) + \frac{i}{6}\int_0^t (R_3((z(\tau) + w^0)^3) - R_3((y(\tau) + v^0)^3)d\tau.$$

Inverting the operator L_{w^0}, $L_{w^0}\varphi(t) = \varphi(t) - \frac{1}{3}B_2(w^0, \varphi(t))$, arguing as above and using the estimate (4.20) for v and w, we obtain on $[0, T^*]$

$$\|\varphi(t)\|_{\dot{H}^s} \le c_7(s)\|L^{-1}_{w^0}\|_{\mathcal{L}(\dot{H}^s)}\big(A\|\varphi(t)\|_{\dot{H}^s} + A\|w^0 - v^0\|_{\dot{H}^s} + M_s^2\int_0^t \|\varphi(\tau)\|_{\dot{H}^s}d\tau\big),$$

where $A = \max(\|y\|_{C([0,T^*];\dot{H}^s)}, \|z\|_{C([0,T^*];\dot{H}^s)})$ and $M_s = M_s(T, \|v^0\|_{\dot{H}^s} + \|w^0\|_{\dot{H}^s})$ is as in (4.20). Hence

$$\|\varphi\|_{C([0,T^*];\dot{H}^s)} \le c_7(s)\|L^{-1}_{w^0}\|_{\mathcal{L}(\dot{H}^s)}\big(A + T^*M_s^2\big)\|\varphi\|_{C([0,T^*];\dot{H}^s)} + C(s, A)\|w^0 - v^0\|_{\dot{H}^s}.$$

Notice that since $w(t)$ conserves energy then by Remark 5.2 we have $\|L^{-1}_{w(t)}\|_{\mathcal{L}(\dot{H}^s)} \le F(\|w^0\|_{\dot{H}^0})$, for all $t \in [0, T]$. Taking A and T^* small enough (both are depending now only on $\|w^0\|_{\dot{H}^s} + \|w^0\|_{\dot{H}^s}$ and T) such that

$$c_7(s)\|L^{-1}_{w^0}\|_{\mathcal{L}(\dot{H}^s)}\big(A + T^*M_s^2\big) \le c_7(s)F(\|w^0\|_{\dot{H}^0})\big(A + T^*M_s^2\big) \le \frac{1}{2}$$

we obtain

$$\|\varphi\|_{C([0,T];\dot{H}^s)} \le 2C(s, A)\|w^0 - v^0\|_{\dot{H}^s} = C'(s, T, \|v^0\|_{\dot{H}^s}, \|w^0\|_{\dot{H}^s})\|w^0 - v^0\|_{\dot{H}^s},$$

which gives for $w(t) - v(t) = \varphi(t) - (w^0 - v^0)$

$$\|w(t) - v(t)\|_{\dot{H}^s} \le \big(C'(s, T, \|v^0\|_{\dot{H}^s}, \|w^0\|_{\dot{H}^s}) + 1\big)\|w^0 - v^0\|_{\dot{H}^s}, \quad t \in [0, T^*].$$

Therefore, after N steps, where $N = [T/T^*] + 1$, we obtain the Lipschitz estimate

$$\|w(t) - v(t)\|_{\dot{H}^s} \le \big(C'(s, T, \|v^0\|_{\dot{H}^s} + \|w^0\|_{\dot{H}^s}) + 1\big)^N\|w^0 - v^0\|_{\dot{H}^s}, \quad t \in [0, T]. \tag{5.18}$$

□

Remark 5.4. Generally speaking the Lipschitz constant in (5.18) may grow with respect to T at a rate higher than exponential. In section 6, and for $s \in [0, 1/2)$, the Lipschitz estimate (5.18) will be proved in a stronger form

$$\|w(t) - v(t)\|_{\dot{H}^s} \le \big(C(s, \|v^0\|_{\dot{H}^0} + \|w^0\|_{\dot{H}^0}) + 1\big)^T\|w^0 - v^0\|_{\dot{H}^s}, \quad t \in [0, T]. \tag{5.19}$$

6. Uniqueness of solutions with non-regular initial data ($0 \le s \le 1/2$) and Lipschitz dependence in weaker norms

Here we will show the uniqueness and Lipschitz continuous dependence on the initial data for the class of solutions of (3.8) in the sense of Definition 3.1 with initial data $v^0 \in \dot{H}^s$ for $s \in [0, 1/2]$. The existence of such solutions, for any $s > 0$, has been established in Theorem 4.3; and we observe that so far we have not proved the *existence* of solutions with initial data in $\dot{H}^0$. The existence and Lipschitz continuity of such solutions, when $s = 0$, will be proved in Theorem 6.4 at the end of this section. We also study the Lipschitz dependence on the initial data in the norm of $\dot{H}^\theta$, for $\theta > -1$ of the solutions of (3.8) with initial data bounded in $\dot{H}^0$.

The main role in the proof is played by *time averaging induced squeezing*, which is described later in this section. First, we give a sketch of the subsequent treatment. Our strategy to prove uniqueness and the Lipschitz continuous dependence of $v \in L_\infty([0, T], \dot{H}^\theta)$,

with $v(0) \in \dot{H}^\theta$, for $\theta \geq 0$, is as follows. Let v, w be two solutions with initial data in $\dot{H}^\theta$ with $v(0) = w(0)$; and let $[0, T_1]$ be the maximal interval on which they coincide (thanks to Remark 3.1 such a maximal interval is closed; also it is possible that $T_1 = 0$). If $T_1 < T$ we may take $v(T_1) = w(T_1) = v^0$ as a new initial data and consider (3.8) on $[T_1, T_1 + \tau]$ with a small τ. Similarly to the proof of Theorem 5.1 we want to transform the problem to an equation in $L_\infty([T_1, T_1 + \tau], \dot{H}^\theta)$ for $y(t)$, where $v(t) = v^0 + y(t)$, of the form

$$y = \mathcal{F}_{\tau,n}(y, v^0), \tag{6.1}$$

where $\mathcal{F}_{\tau,n}$ is a Lipschitz map in $L_\infty([T_1, T_1+\tau], \dot{H}^\theta)$ with a Lipschitz constant less than one. Accordingly, equation (6.1) will have a unique small solution y in $L_\infty([T_1, T_1+\tau], \dot{H}^\theta)$. The parameter n describes the construction of the operator $\mathcal{F}_{\tau,n}$, which involves the splitting of the Fourier modes y_k of the solution y into high modes (with $|k| > n$) and low modes (with $|k| \leq n$). The Lipschitz estimate for $y(t) = v(t) - v^0$ and $z(t) = w(t) - w^0$ will have the form

$$\begin{aligned} &\|\mathcal{F}_{\tau,n}(y, v^0) - \mathcal{F}_{\tau,n}(z, w^0)\|_{L_\infty([T_1,T_1+\tau],\dot{H}^\theta)} \leq \\ &C(C_0)(F_1(n) + \tau F_2(n))\Big(\|y - z\|_{L_\infty([T_1,T_1+\tau],\dot{H}^\theta)} + \|v^0 - w^0\|_{\dot{H}^\theta} \Big), \end{aligned} \tag{6.2}$$

where $C(C_0)$ is bounded provided that the solutions are uniformly bounded, over the interval $[T_1, T_1 + \tau]$, in the $\dot{H}^{\theta_0}$ norm, for some $\theta_0 \geq 0$ (we show below that we can take $\theta_0 = 0$ when $\theta \in (0, 1/2)$)

$$\|v\|_{L_\infty([T_1,T_1+\tau],\dot{H}^{\theta_0})} + \|w\|_{L_\infty([T_1,T_1+\tau],\dot{H}^{\theta_0})} \leq C_0. \tag{6.3}$$

Moreover, the time-independent part of the Lipschitz estimate will enjoy the property

$$F_1\,(n) \to 0 \quad \text{as} \quad n \to \infty. \tag{6.4}$$

Obviously, if we have two solutions y, z in $L_\infty([0, T], \dot{H}^{\theta_0})$, we can take

$$C_0 = C_0(T) = \|v\|_{L_\infty([0,T],\dot{H}^{\theta_0})} + \|w\|_{L_\infty([0,T],\dot{H}^{\theta_0})} < \infty. \tag{6.5}$$

Based on this property, we will first choose n large enough such that

$$C(C_0)F_1(n) \leq 1/4,$$

and then choose τ small enough so that

$$\tau \leq \frac{1}{4C(C_0)F_2(n)}\,.$$

Together, the above implies

$$\|\mathcal{F}_{\tau,n}(y) - \mathcal{F}_{\tau,n}(z)\|_{L_\infty([T_1,T_1+\tau],\dot{H}^\theta)} \leq \frac{1}{2}\|y - z\|_{L_\infty([T_1,T_1+\tau],\dot{H}^\theta)} + C\|v^0 - w^0\|_{\dot{H}^\theta}. \tag{6.6}$$

Therefore the two solutions of (6.1) satisfy the Lipschitz estimate

$$\|y - z\|_{L_\infty([T_1,T_1+\tau],\dot{H}^\theta)} \leq 2C\|v^0 - w^0\|_{\dot{H}^\theta},$$

for some $C = C(T)$, where τ depends on the norm of y, z in $L_\infty([0, T], \dot{H}^{\theta_0})$, and hence for $v(t) = v^0 + y(t)$ and $w(t) = w^0 + z(t)$ this gives

$$\|v - w\|_{L_\infty([T_1,T_1+\tau],\dot{H}^\theta)} \leq (2C(T) + 1)\|v^0 - w^0\|_{\dot{H}^\theta}.$$

If (6.5) holds, we can iterate the estimate and obtain

$$\|v - w\|_{L_\infty([0,T],\dot{H}^\theta)} \leq C'(2C(T) + 1)^{[T/\tau]+1}\|v^0 - w^0\|_{\dot{H}^\theta}. \tag{6.7}$$

Since the $\dot{H}^0$-norm estimates of the solutions constructed in Theorem 4.3 are uniform in T, it follows that in the case when we can take $\theta_0 = 0$, the above Lipschitz continuity estimate can be written in the usual form (see also Remark 5.4)

$$\|v - w\|_{L_\infty([0,T],\dot{H}^\theta)} \le C''(2C+1)^T \|v^0 - w^0\|_{\dot{H}^\theta}. \tag{6.8}$$

Note that two terms in (6.2) control the size of the Lipschitz constant. The first is the Picard short time factor τ, which ensures solvability of ordinary differential equations locally in time. The second term $F_1(n)$ is small thanks to the time averaging induced squeezing which is crucial for the continuous dependence we prove here, and which will be described in detail below.

For the discussion below the reader is referred to the relevant estimates in section 7. Recall that by Corollary 4.1 every solution $v(t)$ of (3.8) in the sense of Definition 3.1 with $v^0 \in \dot{H}^s$, $s > 0$ also satisfies (3.18). Therefore $y(t) = v(t) - v^0$ satisfies (5.15) which we would like to transform to (5.16). Unfortunately, the operator $\int_0^t R_3 dt'$ in (5.15) is Lipschitz in a rather narrow space $\dot{H}^s$, $s > 1/2$, which is out of the range of our interest, namely, $s \in [0, 1/2]$. Therefore we want to use an equation similar to (5.16), but defined in a wider space. A possible candidate is equation (3.31) with the integrated form (3.32). The right-hand side of (3.32) has a small Lipschitz constant in $L_\infty([T_1, T_1+\tau], \dot{H}^\theta)$ if τ is small. But this equation is difficult to use directly for the proof of uniqueness, since the linearization of the left-hand side about $v = v^0$ may be not invertible. But if we try to invert the linearization of the operator $I - B_2$ about v^0 then for $y(t)$, where $v(t) = y(t) + v^0$ with $y(0) = 0$, we obtain from (3.32) the equation

$$\begin{aligned} y(t) - \frac{1}{3}B_2(v^0, y(t)) = \frac{1}{6}B_2(y(t), y(t)) + \frac{1}{18}\Big(B_3\left((y(t)+v^0)^3\right) - B_3\left((v^0)^3\right)\Big) + \\ + \frac{i}{6}\int_0^t \Big(A_{\mathrm{res}}((y(t')+v^0)^3) + \frac{1}{3}B_4((y(t')+v^0)^4)\Big) dt'. \end{aligned}$$

Here we can invert $I - \frac{1}{3}B_2(v^0, y)$ on the left-hand side with norm of the inverse operator depending on $\|v^0\|_{\dot{H}^{\theta_0}}$, where $\theta_0 \ge -1$, see Lemma 5.1, but the Lipschitz constant in $L_\infty([T_1, T_1+\tau], \dot{H}^\theta)$ of the nonlinear operator so obtained is not small because of the second term on the right-hand side (notice that the Lipschitz constant of the first term is small since it is quadratic and y is small, the third term has a small Lipschitz constant because of a short time interval in the integral).

Therefore we derive a modified version of (3.32) (in fact, a family of equations parameterized by $n \in \mathbb{N}$) with the same invertible operator on left-hand side and the second term (corresponding to B_3) having a small Lipschitz constant as $n \to \infty$. To this end we use a proper splitting of a solution into high and low Fourier modes using the projection Π_n defined by (4.2), where we choose n later to be large enough depending on $\|v^0\|_{\dot{H}^{\theta_0}}$.

Now we present a detailed discussion and proofs. We need to introduce some notation. We use projections Π_n with integer $n \ge 0$ defined in (4.2) and we also set

$$\Pi_{-n} = I - \Pi_n.$$

Obviously,

$$u = \Pi_n u + \Pi_{-n} u = \sum_{\zeta = \pm 1} \Pi_{\zeta n} u. \tag{6.9}$$

We can rewrite

$$R_{3\mathrm{nres}}(u,v,w)_k = \sum_{\vec{\zeta} \in \{-1,1\}^3} R_{3\mathrm{nres}}(\Pi_{\vec{\zeta}}(u,v,w))_k, \tag{6.10}$$

where

$$\Pi_{\vec{\zeta}}(u,v,w)=(\Pi_{\zeta_1 n}u,\Pi_{\zeta_2 n}v,\Pi_{\zeta_3 n}w),\quad \vec{\zeta}=(\zeta_1,\zeta_2,\zeta_3)\in\{-1,1\}^3,\quad \zeta_j=\pm1,$$

and where (see (3.20))

$$R_{3\mathrm{nres}}(u,v,w)_k=\sum_{k_1+k_2+k_3=k}^{\mathrm{nonres}}\frac{e^{i3(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1}u_{k_1}v_{k_2}w_{k_3}.$$

Note that in (6.10) the terms with $\zeta_j=+1$ smooth (filter) the $j-$th argument, also the first argument is always smoothed by the factor $1/k_1$. Therefore the only term which has only one smoothed argument (factor) corresponds to $\vec{\zeta}=(\pm1,-1,-1)$, when the already smoothed first factor is multiplied by $\Pi_{\pm n}$ and the remaining two are multiplied by Π_{-n}. The remaining 6 terms in (6.10) have at least two smoothed factors. Hence, we have for $R_{3\mathrm{nres}}(u,v,w)$

$$R_{3\mathrm{nres}}(u,v,w)=R^{(n)}_{3\mathrm{nres}0}(u,v,w)+R^{(n)}_{3\mathrm{nres}1}(u,v,w) \tag{6.11}$$

with

$$\begin{aligned} R^{(n)}_{3\mathrm{nres}0}(u,v,w)&=R_{3\mathrm{nres}}(\Pi_{\vec{\zeta}_0}(u,v,w))+R_{3\mathrm{nres}}(\Pi_{\vec{\zeta}_1}(u,v,w)),\\ R^{(n)}_{3\mathrm{nres}1}(u,v,w)&=\sum_{\vec{\zeta}\in\{-1,1\}^3,\vec{\zeta}\neq\vec{\zeta}_0,\vec{\zeta}\neq\vec{\zeta}_1}R_{3\mathrm{nres}}(\Pi_{\vec{\zeta}}(u,v,w)),\end{aligned} \tag{6.12}$$

where $\vec{\zeta}_0=(+1,-1,-1)$, $\vec{\zeta}_1=(-1,-1,-1)$. Since $\Pi_n+\Pi_{-n}=I$, we obviously have

$$R^{(n)}_{3\mathrm{nres}0}(u,v,w)_k=\sum_{k_1+k_2+k_3=k}^{\mathrm{nonres}}\frac{e^{i3(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1}u_{k_1}\Pi_{-n}v_{k_2}\Pi_{-n}w_{k_3}. \tag{6.13}$$

The operator $R_{3\mathrm{nres}1}$ has better boundedness properties than R_3, as it is shown in the following lemma. However, the corresponding constant is increasing, as $n\to\infty$, and will play the role of the constant $F_2(n)$ in (6.2).

Lemma 6.1. *Let $0\le s\le 1$, $\alpha\ge 0$. Then the operator $R^{(n)}_{3\mathrm{nres}1}$ in (6.12) satisfies the estimate*

$$\|R^{(n)}_{3\mathrm{nres}1}(u,v,w)\|_{\dot H^s}\le c_4 n^{s+1+\alpha}\|u\|_{\dot H^0}\|v\|_{\dot H^{-\alpha}}\|w\|_{\dot H^0}+c_4 n^{1+\alpha}\|u\|_{\dot H^0}\|v\|_{\dot H^{-\alpha}}\|w\|_{\dot H^s}. \tag{6.14}$$

Proof. We consider one of the terms in the second formula in (6.12), namely, the sum of terms with $\vec{\zeta}=(-1,+1,-1)$ and $\vec{\zeta}=(1,+1,-1)$. We set

$$\tilde R^{(n)}_{3\mathrm{nres}1}(u,v,w)_k:=\sum_{k_1+k_2+k_3=k}^{\mathrm{nonres}}\frac{e^{i3(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1}u_{k_1}\Pi_n v_{k_2}\Pi_{-n}w_{k_3};$$

the four remaining terms are estimated in exactly the same way. We use duality

$$|(\tilde R^{(n)}_{3\mathrm{nres}1}(u,v,w),z)|\le\sum_{k\in\mathbb{Z}_0}\sum_{k_1+k_2+k_3=k}^{\mathrm{nonres}}\frac{|u_{k_1}||\Pi_n v_{k_2}||\Pi_{-n}w_{k_3}||z_k|}{|k_1|}\le$$

$$\sum_{k\in\mathbb{Z}_0}\sum_{k_1+k_2+k_3=k}\frac{|u_{k_1}||\Pi_n v_{k_2}||\Pi_{-n}w_{k_3}||z_k|}{|k_1|}=\sum_{k_1,k_2,k_3}\frac{|u_{k_1}||\Pi_n v_{k_2}||\Pi_{-n}w_{k_3}||z_{k_1+k_2+k_3}|}{|k_1|}\le$$

$$\sum_{|k_1|>0}\sum_{0<|k_2|\le n}\sum_{|k_3|>n}\frac{|u_{k_1}||v_{k_2}||w_{k_3}||z_{k_1+k_2+k_3}|}{|k_1|}.$$

For $0 \le s \le 1$ we set $\tilde z_k = |z_k|/|k|^s$ and obtain

$$|(\tilde R^{(n)}_{3\text{nres}1}(u,v,w),z)| \le \sum_{|k_1|>0}\sum_{0<|k_2|\le n}\sum_{|k_3|>n}\frac{|u_{k_1}||v_{k_2}||w_{k_3}||k_1+k_2+k_3|^s|\tilde z_k|}{|k_1|} \le$$

$$\sup_{0<|k_2|\le n}|v_{k_2}|\sum_{0<|k_2|\le n}\sum_{|k_1|>0}\sum_{|k_3|>n}\frac{|u_{k_1}||w_{k_3}|(|k_1|^s+|k_2|^s+|k_3|^s)\tilde z_{k_1+k_2+k_3}}{|k_1|} \le$$

$$\|v\|_{\dot H^{-\alpha}}n^{\alpha}\sum_{0<|k_2|\le n}\sum_{|k_1|>0}\sum_{|k_3|>n}\frac{|u_{k_1}||w_{k_3}|(|k_1|^s+|k_2|^s+|k_3|^s)\tilde z_{k_1+k_2+k_3}}{|k_1|}.$$

We estimate the sum in k_1 and k_3 as follows:

$$\sum_{|k_1|>0}\sum_{|k_3|>n}\frac{|u_{k_1}||w_{k_3}|(|k_1|^s+|k_2|^s+|k_3|^s)\tilde z_{k_1+k_2+k_3}}{|k_1|} \le$$

$$n^s\sum_{|k_1|>0}\sum_{|k_3|>n}\frac{|u_{k_1}||w_{k_3}|\tilde z_{k_1+k_2+k_3}}{|k_1|} + \sum_{|k_1|>0}\sum_{|k_3|>n}\frac{|u_{k_1}||w_{k_3}||k_1|^s\tilde z_{k_1+k_2+k_3}}{|k_1|} +$$

$$\sum_{|k_1|>0}\sum_{|k_3|>n}\frac{|u_{k_1}||w_{k_3}||k_3|^s\tilde z_{k_1+k_2+k_3}}{|k_1|} \le$$

$$n^s\left(\sum_{k_1}\frac{1}{|k_1|^2}\sum_{k_3}\tilde z^2_{k_1+k_2+k_3}\right)^{1/2}\left(\sum_{k_1}\sum_{k_3}|u_{k_1}|^2|w_{k_3}|^2\right)^{1/2} +$$

$$2\left(\sum_{k_1}\frac{1}{|k_1|^2}\sum_{k_3}\tilde z^2_{k_1+k_2+k_3}\right)^{1/2}\left(\sum_{k_1}\sum_{k_3}|w_{k_3}|^2|k_3|^{2s}|v_{k_1}|^2\right)^{1/2} \le$$

$$c_3 n^s\|v\|_{\dot H^0}\|w\|_{\dot H^0}\|z\|_{\dot H^{-s}} + 2c_3\|v\|_{\dot H^0}\|w\|_{\dot H^s}\|z\|_{\dot H^{-s}},$$

where $c_3 \le \pi/\sqrt{3}$. Hence, after a finite summation in k_2 we obtain

$$|(\tilde R^{(n)}_{3\text{nres}1}(u,v,w),z)| \le$$
$$c_4 n^{s+1+\alpha}\|u\|_{\dot H^0}\|v\|_{\dot H^{-\alpha}}\|w\|_{\dot H^0}\|z\|_{\dot H^{-s}} + c_4 n^{1+\alpha}\|u\|_{\dot H^0}\|v\|_{\dot H^{-\alpha}}\|w\|_{\dot H^s}\|z\|_{\dot H^{-s}}$$

and (6.14) is proven. □

We now consider the operator $R^{(n)}_{3\text{nres}0}$ defined in (6.13):

$$R^{(n)}_{3\text{nres}0}(v^3)_k = \sum_{k_1+k_2+k_3=k}^{\text{nonres}}\frac{e^{i3(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1}v_{k_1}\Pi_{-n}v_{k_2}\Pi_{-n}v_{k_3}.$$

We make the transformation of the form (3.24)–(3.29) (second differentiation by parts in time) applied to $R^{(n)}_{3\text{nres}0}$ instead of $R_{3\text{nres}}$. Namely, transformation (3.24) is replaced by

$$R^{(n)}_{3\text{nres}0}(v^3)_k = \frac{1}{3i}\partial_t B^{(n)}_{30}(v,v,v)_k - \frac{1}{3i}\sum_{k_1+k_2+k_3=k}^{\text{nonres}}\frac{e^{i3(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1(k_1+k_2)(k_2+k_3)(k_3+k_1)}\times$$
$$\big(\partial_t v_{k_1}\Pi_{-n}v_{k_2}\Pi_{-n}v_{k_3} + v_{k_1}\Pi_{-n}\partial_t v_{k_2}\Pi_{-n}v_{k_3} + v_{k_1}\Pi_{-n}v_{k_2}\Pi_{-n}\partial_t v_{k_3}\big).$$

Similarly to (3.25), the operator $B^{(n)}_{30}$ is given by

$$B^{(n)}_{30}(v,v,v)_k = \sum_{k_1+k_2+k_3=k}^{\text{nonres}}\frac{e^{i3(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1(k_1+k_2)(k_2+k_3)(k_3+k_1)}v_{k_1}\Pi_{-n}v_{k_2}\Pi_{-n}v_{k_3}. \tag{6.15}$$

We obtain similarly to (3.30)

$$R^{(n)}_{3\text{nres}0}(v^3)_k = \frac{1}{3i}\partial_t B^{(n)}_{30}(v^3)_k - \frac{1}{3}B^{(n)}_{40}(v^4)_k, \tag{6.16}$$

with

$$B^{(n)}_{40}(v^4)_k = \frac{1}{2}B^1_{40}(v^4)_k + B^2_{40}(v^4)_k,$$

where $B^1_{40} = B^{1(n)}_{40}$, $B^2_{40} = B^{2(n)}_{40}$ are defined similarly to (3.28), (3.29):

$$B^1_{40}(v^4)_k = \sum^{\text{nonres}}_{k_1+k_2+k_3+k_4=k} \frac{e^{i\Phi(k_1,k_2,k_3,k_4)t}}{(k_1+k_2)(k_1+k_3+k_4)(k_2+k_3+k_4)}\Pi_{-n}v_{k_1}\Pi_{-n}v_{k_2}v_{k_3}v_{k_4}, \tag{6.17}$$

$$B^2_{40}(v^4)_k = \sum^{\text{nonres}}_{k_1+k_2+k_3+k_4=k} \frac{e^{i\Phi(k_1,k_2,k_3,k_4)t}\ (k_3+k_4)}{k_1(k_1+k_2)(k_1+k_3+k_4)(k_2+k_3+k_4)}v_{k_1}\Pi_{-n}v_{k_2}\Pi_{-n}\left(v_{k_3}v_{k_4}\right). \tag{6.18}$$

In the last formula the operator $\Pi_{-n}(v_{k_3}v_{k_4})$ is defined as follows:

$$\Pi_{-n}(v_{k_3}v_{k_4}) = \begin{cases} v_{k_3}v_{k_4} & \text{if } |k_3+k_4| > n \\ 0 & \text{if } |k_3+k_4| \le n \end{cases}.$$

Now, similarly to (3.31), based on (6.11) and (6.16), we obtain from (3.15) for every fixed $n \in \mathbb{N}$ the following family of equations, which we call *the third form of the KdV*:

$$\partial_t\left(v_k - \frac{1}{6}B_2(v^2)_k - \frac{1}{18}B^{(n)}_{30}(v^3)_k\right) = \frac{i}{6}R_{3\text{res}}(v^3)_k + \frac{i}{6}R^{(n)}_{3\text{nres}1}(v^3)_k + \frac{i}{18}B^{(n)}_{40}(v^4)_k, \tag{6.19}$$

where $k \in \mathbb{Z}_0$. Integrating (6.19) we obtain

$$\begin{aligned}\left(v_k - \frac{1}{6}B_2(v^2)_k - \frac{1}{18}B^{(n)}_{30}(v^3)_k\right)(t) - \left(v_k - \frac{1}{6}B_2(v^2)_k - \frac{1}{18}B^{(n)}_{30}(v^3)_k\right)(0) = \\ \frac{i}{6}\int_0^t\left(R_{3\text{res}}(v^3)_k + R^{(n)}_{3\text{nres}1}(v^3)_k + \frac{1}{3}B^{(n)}_{40}(v^4)_k\right)(t')dt'.\end{aligned} \tag{6.20}$$

For $v(t) = y(t) + v^0$, $y(0) = 0$, we obtain as in (5.15) the equivalent equation

$$\begin{aligned}y(t) - \frac{1}{3}B_2(v^0, y(t)) = \frac{1}{6}B_2(y(t), y(t)) + \frac{1}{18}\left(B^{(n)}_{30}\left((y(t)+v^0)^3\right) - B^{(n)}_{30}\left((v^0)^3\right)\right) + \\ +\frac{i}{6}\int_0^t\left(R_{3\text{res}}((y(t')+v^0)^3) + R^{(n)}_{3\text{nres}1}((y(t')+v^0)^3) + \frac{1}{3}B^{(n)}_{40}((y(t')+v^0)^4)\right)dt'.\end{aligned} \tag{6.21}$$

Theorem 6.1. *Equation (6.21) can be written in the form (6.1) with $\mathcal{F}_{\tau,n}(y, v^0)$ satisfying (6.2) and (6.4) with $\theta \in [0, 1/2]$. Here $\theta_0 = 0$ for $\theta < 1/2$ and $\theta_0 > 0$ for $\theta = 1/2$.*

Proof. We apply the linear operator $L^{-1}_{v^0}$, which is inverse to $I - \frac{1}{3}B_2(v^0, y)$, to (6.21) and obtain similarly to (5.16) the equation (6.1). The Lipschitz estimate (6.2) and the fact that $C(C_0)$ is bounded if (6.3) holds follow from the estimate $\|L^{-1}_{v^0}\|_{\dot H^\theta \to \dot H^\theta} \le F(\|v^0\|_{\dot H^0})$, $\theta \ge 0$ (see Lemma 5.1, where, in fact, $\theta \ge -1$) and the boundedness of the multilinear operators B^1_{40}, B^2_{40}, $R_{3\text{res}} = A_{\text{res}}$, $R^{(n)}_{3\text{nres}1}$ in $\dot H^s$, see Lemma 7.12, Lemma 7.13, Lemma 7.14 and Lemma 6.1 (with $\alpha = 0$). The estimate (6.4) follows from Lemma 7.10. ☐

Theorem 6.2. *Let $0 \le s \le 1/2$. Let $v(0) = v^0 \in \dot{H}^s$ be given. Then there exists n_0 large enough and T^* small enough, both depending on $\|v^0\|_{\dot{H}^0}$, such that for each $n \ge n_0$ the equation (6.21) has a unique local solution $y \in C_0([0,T^*]; \dot{H}^s)$ and $v = y + v^0 \in C([0,T^*]; \dot{H}^s)$ is a unique local solution of (6.20) on the time interval $[0,T^*]$ with sufficiently small T^*. This solution $v(t)$ can be extended to the maximal interval $[0, T^*_{\max})$ such that either $T^*_{\max} = +\infty$ or $\limsup_{t \to T^*_{\max} - 0} \|v(t)\|_{\dot{H}^s} = +\infty$.*

Proof. The proof is similar to the proof of the first part of Theorem 5.1. We use the Contraction Principle. Based on (6.2) and (6.4) we derive (6.6) for $[0,T]$ with T small enough and n large enough. The existence and uniqueness of a solution to (6.21) follows in a standard way from the Contraction Principle. Extension to the maximal interval is treated as in Theorem 5.1, in particular, the uniqueness of a solution to (6.20) on a maximal interval follows in a standard way from the proof of the local uniqueness. □

Remark 6.1. Analogous local existence and uniqueness theorem for solutions of (6.20) (similar results can be found in [7], [8]) can be proven for $0 \ge s > -1$ using the boundedness in negative spaces of linear operator $L^{-1}_{v^0}$ and multilinear operators which enter (6.20). We do not consider this case here.

Theorem 6.3. *Let $s \in (0, 1/2]$ and $T > 0$. For any initial data $v(0) = v^0 \in \dot{H}^s$ the solution $v = v^\infty$ of equation (3.9), constructed in Theorem 4.3, is unique on the interval $t \in [0,T]$, is of the class $C([0,T], \dot{H}^s)$ and depends Lipschitz continuously on the initial data in the sense of (6.7).*

Proof. First we observe that for each $n \in \mathbb{N}$ one can construct a Galerkin truncated version of (6.19). In fact, this can be done step by step as the previous derivation of (6.19) but starting from (4.7). Therefore it is clear that the Galerkin solution of equation (4.1) (and (4.7)) established in Proposition 4.1 is also the unique solution of the Galerkin version of (6.20) for every $n \in \mathbb{N}$. Notice that $v^{(m)}$ is independent of n. Similarly to Theorem 4.3 we can pass to the limit along a subsequence $m_j \to \infty$ in the Galerkin version of the equation (6.20) (as in Corollary 4.1) to see that the solution $v = v^\infty$ of (3.9) also satisfies (6.20) for each n. Therefore $v = v^\infty$ belongs to $C([0, T^*_{\max}), \dot{H}^s)$ on the maximal interval of existence in Theorem 6.2 and is unique. In addition, $\|v(t)\|_{\dot{H}^0} = \|v(0)\|_{\dot{H}^0}$ for all $t \in [0, T^*_{\max})$. From Theorem 4.3 we have that v is bounded in $\dot{H}^s$ on any finite time interval. Hence $T^*_{\max} = \infty$, the energy is conserved for all t and $v = v^\infty \in C([0,T], \dot{H}^s)$ for any finite T. Finally, the Lipschitz continuity follows from (6.7) with $\theta = s$. □

Finally, we consider the case $s = 0$, which still remains unsettled. For this purpose we use the regularization of the initial data.

Theorem 6.4. *Let $s = 0$ and $T > 0$. For any initial data $v(0) = v^0 \in \dot{H}^0$ the equation (3.9) has on $t \in [0,T]$ a unique solution of class $C([0,T], \dot{H}^0)$, which depends Lipschitz continuously on the initial data.*

Proof. Let $v(0) = v^0 \in \dot{H}^0$. We approximate the initial data v^0 by a sequence of smooth functions $v^0_{(j)} \in \dot{H}^{s_0}$, where $s_0 > 0$, and $\|v^0 - v^0_{(j)}\|_{\dot{H}^0} \to 0$ as $j \to \infty$. By Theorem 6.3, for each j there exits a unique global solution $v_{(j)} \in C([0,T], \dot{H}^{s_0})$, which conserves energy, that is, $\|v_{(j)}(t)\|_{\dot{H}^0} = \|v_{(j)}(0)\|_{\dot{H}^0}$. Moreover, from the Lipschitz estimate (6.7) we have

$$\|v_{(j)} - v_{(i)}\|_{L_\infty([0,T],\dot{H}^0)} \le C(T, \|v^0\|_{\dot{H}^0}) \|v^0_{(j)} - v^0_{(i)}\|_{\dot{H}^0}.$$

Since $v^0_{(j)}$ is a Cauchy sequence in $\dot{H}^0$, it follows that $v_{(j)}(t)$ is a Cauchy sequence in $C([0,T], \dot{H}^0)$. We again denote the corresponding limit by $v^\infty(t)$, although the origin

of this limit is different from that in Theorem 4.3. We can pass to the limit in equation (3.9) for $v = v_{(j)}$, as $j \to \infty$, in a similar (simpler) way as we did in Theorem 4.3. We obtain that v^∞ is a solution of (3.9), which conserves energy since $\|v^\infty(t)\|_{\dot H^0} = \lim_{j\to\infty} \|v_{(j)}(t)\|_{\dot H^0} = \lim_{j\to\infty} \|v^0_{(j)}\|_{\dot H^0} = \|v^0\|_{\dot H^0}$, for all $t \in [0,T]$. A totally similar passage to the limit shows that v^∞ is a solution of (3.18), (3.32), and, most importantly, of equation (6.20) for any $n \in \mathbb{N}$. Equation (6.20) for a sufficiently large n depending on $\|v^0\|_{\dot H^0}$ has a unique local solution in view of Theorem 6.2. However, there we have $T^*_{\max} = \infty$, since $\|v^\infty(t)\|_{\dot H^0} = \|v^\infty(0)\|_{\dot H^0}$. Since v^∞ is the unique solution of (6.20) for n large enough, we apply Theorem 6.2 to show the Lipschitz continuous dependence on the initial data in the $\dot H^0$ norm. □

We now study the Lipschitz dependence of $v\,(t)$ on the initial data $v(0)$ in $\dot H^s$, $s < 0$ if $v\,(0)$ is bounded in $\dot H^0$.

Theorem 6.5. *Let $T > 0$ be given, and $\theta \in (-1, 0]$. If $\|v^0\|_{\dot H^0} + \|w^0\|_{\dot H^0} \le M$, then the two solutions $v(t)$ and $w(t)$ of equation (3.9), with $v(0) = v^0$ and $w(0) = w^0$, satisfy the estimate*

$$\|v - w\|_{L_\infty([0,T],\dot H^\theta)} \le C' C_M^T \|v^0 - w^0\|_{\dot H^\theta}, \tag{6.22}$$

where C', C_M depend only on M and θ.

Proof. We derive (6.6) and (6.7) based on the boundedness in negative spaces of linear operator $L^{-1}_{v^0}$ and multilinear operators which enter (6.20). Namely, we use Lemma 7.5, Lemma 7.10, Lemma 7.12 and Lemma 7.13. The intersection of the range of the values of s in the above lemmas and Lemma 5.1 gives the restriction on the range of θ, namely $\theta \in (-1, 0]$. From (6.6) we infer (6.7) which yields (6.22). □

7. Appendix. Estimates for convolution operators

In this section we study the continuity and smoothing properties of the convolution operators B_1, B_2, B_3, $B^{(n)}_{30}$, B_4, $B^{(n)}_{40}$, B^1_{40}, B^2_{40}, R_3 and the resonant operator A_{res} in the Sobolev spaces $\dot H^s$. Although the corresponding constants c_k do not play a role in our analysis, the explicit expressions for them can easily be given and, in fact, are given in most cases at the end of the proofs.

Lemma 7.1. *Let $\theta > 3/2$. Then the bilinear operator B_1 defined in (3.11) maps $\dot H^0 \times \dot H^0$ into $\dot H^{-\theta}$ and satisfies the estimate*

$$\|B_1(u, v)\|_{\dot H^{-\theta}} \le c_1(\theta) \|u\|_{\dot H^0} \|v\|_{\dot H^0}. \tag{7.1}$$

Proof. We observe that setting $d_k = \sum_{k_1+k_2=k} e^{i3kk_1k_2t} u_{k_1} v_{k_2}$ we have $|d_k| \le \|u\|_{\dot H^0}\|v\|_{\dot H^0}$. Furthermore, for $\sigma > 1/2$ we see that $d \in \dot H^{-\sigma}$ with

$$\|d\|_{\dot H^{-\sigma}} = \left(\sum_{k\in\mathbb{Z}_0} |d_k|^2 k^{-2\sigma}\right)^{1/2} \le c(\sigma)\|u\|_{\dot H^0}\|v\|_{\dot H^0} \quad \text{and} \quad c(\sigma) = \left(\sum_{k\in\mathbb{Z}_0} |k|^{-2\sigma}\right)^{1/2}.$$

Hence for $\theta = \sigma + 1 > 3/2$, $B_1(u, v) \in \dot H^{-\theta}$ with $\|B_1(u, v)\|_{\dot H^{-\theta}} \le \frac{1}{2} c(\theta - 1)\|u\|_{\dot H^0}\|v\|_{\dot H^0}$. □

Lemma 7.2. *Let $s > -1/2$. Then the bilinear operator B_2 defined in (3.16) maps $\dot H^s \times \dot H^s$ into $\dot H^{s+1}$ and satisfies the estimate*

$$\|B_2(u, v)\|_{\dot H^{s+1}} \le c_2(s + 1)\|u\|_{\dot H^s}\|v\|_{\dot H^s}. \tag{7.2}$$

Proof. By duality, (7.2) is equivalent to the estimate

$$|(B_2(u,v),z)| \le c_2(s+1)\|u\|_{\dot H^s}\|v\|_{\dot H^s}\|z\|_{\dot H^{-(s+1)}}, \tag{7.3}$$

where z is an arbitrary element in $\dot H^{-(s+1)}$. Replacing the exponentials in (3.16) by unity, setting below $\tilde u_k = |u_k||k|^s$, $\tilde v_k = |v_k||k|^s$, $\tilde z_k = |z_k||k|^{-s-1}$, and using the inequality $|k_1+k_2|^{s+1} \le 2^s(|k_1|^{s+1}+|k_2|^{s+1})$, we have

$$|(B_2(u,v),z)| \le \sum_k \sum_{k_1+k_2=k} \frac{|u_{k_1}||v_{k_2}||z_k|}{|k_1||k_2|} =$$

$$\sum_{k_1}\sum_{k_2} \frac{|u_{k_1}||v_{k_2}||z_{k_1+k_2}|}{|k_1||k_2|} = \sum_{k_1}\sum_{k_2} \frac{\tilde u_{k_1}\tilde v_{k_2}\tilde z_{k_1+k_2}|k_1+k_2|^{1+s}}{|k_1|^{1+s}|k_2|^{1+s}} \le$$

$$2^s\sum_{k_1}\sum_{k_2}\frac{\tilde u_{k_1}\tilde v_{k_2}\tilde z_{k_1+k_2}}{|k_2|^{1+s}} + 2^s\sum_{k_1}\sum_{k_2}\frac{\tilde u_{k_1}\tilde v_{k_2}\tilde z_{k_1+k_2}}{|k_1|^{1+s}} \le$$

$$2^s\left(\sum_{k_1}\sum_{k_2}\tilde u_{k_1}^2\tilde v_{k_2}^2\right)^{1/2}\left(\sum_{k_2}\frac{1}{|k_2|^{2s+2}}\sum_{k_1}\tilde z_{k_1+k_2}^2\right)^{1/2} +$$

$$2^s\left(\sum_{k_1}\sum_{k_2}\tilde u_{k_1}^2\tilde v_{k_2}^2\right)^{1/2}\left(\sum_{k_1}\frac{1}{|k_1|^{2s+2}}\sum_{k_2}\tilde z_{k_1+k_2}^2\right)^{1/2} =$$

$$2^{s+1}c(s+1)\|u\|_{\dot H^s}\|v\|_{\dot H^s}\|z\|_{\dot H^{-(s+1)}},$$

where

$$c(p) = \Big(\sum_{j\in\mathbb{Z}_0} j^{-2p}\Big)^{1/2}, \quad p > 1/2. \tag{7.4}$$

This proves (7.3), (7.2) and provides an expression for the constant $c_2(s+1)$. □

Lemma 7.2 essentially proves the following well-known Banach algebra property of the Sobolev spaces H^s, $s > 1/2$.

Corollary 7.1. *If $u, v \in H^s$, $s > 1/2$, then the product $uv \in H^s$ and*

$$\|uv\|_{H^s} \le K_1(s)\|u\|_{H^s}\|v\|_{H^s}. \tag{7.5}$$

Proof. The estimate (7.5) is equivalent to

$$|(uv,z)| \le K_1(s)\|u\|_{H^s}\|v\|_{H^s}\|z\|_{H^{-s}}. \tag{7.6}$$

Setting $\tilde u_k = |u_k||k|^s$, $\tilde v_k = |v_k||k|^s$, $\tilde z_k = |z_k||k|^{-s}$ for $k \in \mathbb{Z}_0$ and arguing as in Lemma 7.2 we have

$$|(uv,z)| \le \sum_{k_1\in\mathbb{Z}}\sum_{k_2\in\mathbb{Z}} |u_{k_1}||v_{k_2}||z_{k_1+k_2}| = |z_0|\sum_{k_1+k_2=0}|u_{k_1}||v_{k_2}| + |u_0|\sum_{k_2\in\mathbb{Z}_0}\tilde v_{k_2}\tilde z_{k_2} +$$

$$|v_0|\sum_{k_1\in\mathbb{Z}_0}\tilde u_{k_1}\tilde z_{k_1} + \sum_{k_1\in\mathbb{Z}_0}\sum_{k_2\in\mathbb{Z}_0}\frac{\tilde u_{k_1}\tilde v_{k_2}|k_1+k_2|^s\tilde z_{k_1+k_2}}{|k_1|^s|k_2|^s} \le (3+2^sc(s-1))\|u\|_{H^s}\|v\|_{H^s}\|z\|_{H^{-s}},$$

which proves (7.6) providing an expression for $K_1(s)$. □

Lemma 7.3. *Let $s+\alpha \ge 0$, $\alpha < 3/4$, $s > -3/4$. Then the bilinear operator B_2 defined in (3.16) maps $\dot H^s \times \dot H^s$ into $\dot H^{s+\alpha}$ and satisfies the estimate*

$$\|B_2(u,v)\|_{\dot H^{s+\alpha}} \le c_2'(s,\alpha)\|u\|_{\dot H^s}\|v\|_{\dot H^s}. \tag{7.7}$$

Proof. By duality, (7.7) is equivalent to the estimate

$$|(B_2(u,v),z)| \le c_2'(s,\alpha)\|u\|_{\dot H^s}\|v\|_{\dot H^s}\|z\|_{\dot H^{-(s+\alpha)}},$$

where z is an arbitrary element in $\dot H^{-(s+\alpha)}$. Replacing the exponentials in (3.16) by unity, setting below $\tilde u_k = |u_k||k|^s$, $\tilde v_k = |v_k||k|^s$, $\tilde z_k = |z_k||k|^{-s-\alpha}$, and using the inequality $|k_1+k_2|^{s+\alpha} \le b(|k_1|^{s+\alpha}+|k_2|^{s+\alpha})$, with $b = \max(2^{s+\alpha-1},1)$ we have

$$|(B_2(u,v),z)| \le \sum_k \sum_{k_1+k_2=k} \frac{|u_{k_1}||v_{k_2}||z_k|}{|k_1||k_2|} =$$

$$\sum_{k_1}\sum_{k_2} \frac{|u_{k_1}||v_{k_2}||\tilde z_{k_1+k_2}||k_1+k_2|^{\alpha+s}}{|k_1||k_2|} \le b\sum_{k_1}\sum_{k_2} \frac{\tilde u_{k_1}\tilde v_{k_2}(|k_1|^{s+\alpha}+|k_2|^{s+\alpha})\tilde z_{k_1+k_2}}{|k_1|^{1+s}|k_2|^{1+s}} \le$$

$$b\left(\sum_{k_1}\sum_{k_2}\tilde u_{k_1}^2\frac{\tilde z_{k_1+k_2}}{|k_2|^{2-2\alpha}}\right)^{1/2}\left(\sum_{k_2}\sum_{k_1}\tilde v_{k_2}^2\frac{\tilde z_{k_1+k_2}}{|k_1|^{2+2s}}\right)^{1/2} +$$

$$b\left(\sum_{k_1}\sum_{k_2}\tilde u_{k_1}^2\frac{\tilde z_{k_1+k_2}}{|k_2|^{2s+2}}\right)^{1/2}\left(\sum_{k_2}\sum_{k_1}\tilde v_{k_2}^2\frac{\tilde z_{k_1+k_2}}{|k_1|^{2-2\alpha}}\right)^{1/2},$$

where $4-4\alpha > 1$. Note that

$$\sum_{k_2}\frac{|\tilde z_{k_1+k_2}|}{|k_2|^p} \le \left(\sum_{k_2}|\tilde z_{k_1+k_2}|^2\right)^{1/2}\left(\sum_{k_2}\frac{1}{|k_2|^{2p}}\right)^{1/2} \le c(p)\|\tilde z\|_{\dot H^0}, \quad 2p>1$$

with the same $c(p)$ as in (7.4). Since $2-2\alpha > 1/2$ and $2+2s > 1/2$, we have

$$|(B_2(u,v),z)| \le 2b\,c(2-2\alpha)c(2+2s)\|\tilde u\|_{\dot H^0}\|\tilde v\|_{\dot H^0}\|\tilde z\|_{\dot H^0} = c_2'(s,\alpha)\|u\|_{\dot H^s}\|v\|_{\dot H^s}\|z\|_{\dot H^{-(s+\alpha)}}.$$

□

Lemma 7.4. *Let $u,v \in H^{-1}$ and $S > 1/2$. Then*

$$\|B_2(u,v)\|_{\dot H^{-S}} \le c(S)\|u\|_{\dot H^{-1}}\|v\|_{\dot H^{-1}}. \tag{7.8}$$

Proof. Arguing by duality, we take an arbitrary element z in $\dot H^S$ and estimate the inner product:

$$|(B_2(u,v),z)| \le \sum_k\sum_{k_1+k_2=k}\frac{|u_{k_1}||v_{k_2}||z_k|}{|k_1||k_2|} = \sum_{k_1}\sum_{k_2}\frac{|u_{k_1}||v_{k_2}||z_{k_1+k_2}|}{|k_1||k_2|} \le$$

$$\left(\sum_{k_1}\sum_{k_2}\frac{|u_{k_1}|^2}{|k_1|^2}|z_{k_1+k_2}|\right)^{1/2}\left(\sum_{k_1}\sum_{k_2}\frac{|v_{k_2}|^2|z_{k_1+k_2}|}{|k_2|^2}\right)^{1/2} = \|u\|_{\dot H^{-1}}\|v\|_{\dot H^{-1}}\sum_{k\in\mathbb{Z}_0}|z_k|.$$

Since

$$\sum_{k\in\mathbb{Z}_0}|z_k| \le \left(\sum_{k\in\mathbb{Z}_0}|z_k|^2|k|^{2S}\right)^{1/2}\left(\sum_{k\in\mathbb{Z}_0}|k|^{-2S}\right)^{1/2} = c(S)\|z\|_{\dot H^S}, \tag{7.9}$$

this proves (7.8). □

Lemma 7.5. *Let $-7/4 < s \le 0$. Then the bilinear operator B_2 defined in (3.16) maps $\dot H^0 \times \dot H^s$ into $\dot H^s$, and satisfies the estimate*

$$\|B_2(u,v)\|_{\dot H^s} \le c_2''(s)\|u\|_{\dot H^0}\|v\|_{\dot H^s}. \tag{7.10}$$

Proof. Using duality, it is sufficient to estimate

$$|B_2(u,v),z)| \le c_2''(s)\|u\|_{\dot H^0}\|v\|_{\dot H^s}\|z\|_{\dot H^{-s}},$$

where z is an arbitrary element in $\dot H^{-s}$. We set $p=-s$, $0\le p<7/4$. Setting below $\tilde u_k=|u_k|$, $\tilde v_k=|v_k||k|^s$, $\tilde z_k=|z_k||k|^{-s}=|z_k||k|^p$, we have

$$|B_2(u,v),z)| \le \sum_{k\in\mathbb{Z}_0}\sum_{k_1+k_2=k}\frac{|u_{k_1}||v_{k_2}||z_k|}{|k_1||k_2|} = \sum_{k_1}\sum_{k_2}\frac{|u_{k_1}||v_{k_2}||z_{k_1+k_2}|}{|k_1||k_2|} =$$

$$\sum_{k_1,k_2}\frac{\tilde u_{k_1}\tilde v_{k_2}\tilde z_{k_1+k_2}}{|k_1||k_1+k_2|^p|k_2|^{1-p}} = \sum_{k_1,k_2}\frac{(\tilde u_{k_1}\tilde v_{k_2})^{1/2}(\tilde v_{k_2}\tilde z_{k_1+k_2})^{1/2}(\tilde u_{k_1}\tilde z_{k_1+k_2})^{1/2}}{|k_1||k_1+k_2|^p|k_2|^{1-p}} \le$$

$$\left(\sum_{k_1,k_2}(\tilde u_{k_1}\tilde v_{k_2})^2\right)^{1/4}\left(\sum_{k_1,k_2}(\tilde v_{k_2}\tilde z_{k_1+k_2})^2\right)^{1/4}\left(\sum_{k_1,k_2}(\tilde u_{k_1}\tilde z_{k_1+k_2})^2\right)^{1/4}\left(\sum_{k_1,k_2}K_2^4\right)^{1/4} =$$

$$\|u\|_{\dot H^0}\|v\|_{\dot H^s}\|z\|_{\dot H^{-s}}\left(\sum_{k_1,k_2}K_2^4\right)^{1/4},$$

where

$$\sum_{k_1,k_2}K_2^4 = \sum_{k_1,k_2}\frac{1}{|k_1|^4|k_1+k_2|^{4p}|k_2|^{4-4p}} = \sum_{k_1}\frac{1}{|k_1|^4}\sum_{k_2}\frac{1}{|k_1+k_2|^{4p}|k_2|^{4-4p}} =: (c_2''(s))^4.$$

If $p\le 1$, then $c_2''(s)\le(\sum_{k\in\mathbb{Z}_0}|k|^{-4})^{1/2}=c(2)$, since

$$\sum_{k_2}\frac{1}{|k_1+k_2|^{4p}|k_2|^{4-4p}} \le \left(\sum_{k_2}\frac{1}{|k_1+k_2|^4}\right)^p\left(\sum_{k_2}\frac{1}{|k_2|^4}\right)^{1-p} = \sum_{k\in\mathbb{Z}_0}|k|^{-4}.$$

If $7/4>p>1$ (so that $4p-4<3$ and $4p-8<-1$) we have (see (7.4))

$$\sum_{k_1,k_2}K_2^4 \le \sum_{k_1,k_2}\frac{|k_2|^{4p-4}}{|k_1|^4|k_1+k_2|^{4p}} \le 8\sum_{k_1,k_2}\frac{|k_1|^{4p-4}+|k_1+k_2|^{4p-4}}{|k_1|^4|k_1+k_2|^{4p}} =$$

$$8\sum_{k_1,k_2}\frac{|k_1|^{4p-8}}{|k_1+k_2|^{4p}} + 8\sum_{k_1,k_2}\frac{1}{|k_1|^4|k_1+k_2|^4} = 8c(4-2p)^2c(2p)^2+8c(2)^4,$$

which gives in this case $c_2''(s)\le\big(8c(4-2p)^2c(2p)^2+8c(2)^4\big)^{1/4}$. □

Lemma 7.6. *Let $s\ge 0$. Then the trilinear operator B_3 in (3.25) maps $(\dot H^s)^3$ into $\dot H^{s+2}$ and satisfies the estimate*

$$\|B_3(u,v,w)\|_{\dot H^{s+2}} \le c_3(s)\|u\|_{\dot H^s}\|v\|_{\dot H^s}\|w\|_{\dot H^s}. \tag{7.11}$$

Proof. Arguing by duality we start with the case $s=0$. Setting $\tilde z_k=|z_k|/k^2$ and using the inequality $|k_1+k_2+k_3|\le|k_1|+|k_2+k_3|$ we have

$$|(B_3(u,v,w),z)| \le \sum_{k_1,k_2,k_3}^{\text{nonres}}\frac{|u_{k_1}||v_{k_2}||w_{k_3}|\,\tilde z_{k_1+k_2+k_3}\,|k_1+k_2+k_3|^2}{|k_1||k_1+k_2||k_2+k_3||k_3+k_1|} \le$$

$$\sum_{k_1,k_2,k_3}^{\text{nonres}}\frac{|u_{k_1}||v_{k_2}||w_{k_3}|\,\tilde z_{k_1+k_2+k_3}\,|k_1+k_2+k_3|}{|k_1+k_2||k_2+k_3||k_3+k_1|} + \sum_{k_1,k_2,k_3}^{\text{nonres}}\frac{|u_{k_1}||v_{k_2}||w_{k_3}|\,\tilde z_{k_1+k_2+k_3}\,|k_1+k_2+k_3|}{|k_1||k_1+k_2||k_3+k_1|}.$$

Next, using the inequalities

$$|k_1+k_2+k_3| \le \begin{cases} \frac{1}{2}|k_1+k_2|+\frac{1}{2}|k_2+k_3|+\frac{1}{2}|k_3+k_1|, \\ |k_1|+|k_1+k_2|+|k_3+k_1|, \end{cases} \tag{7.12}$$

in the first and second sum, respectively, we continue

$$|(B_3(u,v,w),z)| \le \sum_{k_1,k_2,k_3}^{\text{nonres}} |u_{k_1}||v_{k_2}||w_{k_3}|\, \tilde{z}_{k_1+k_2+k_3} \times$$
$$\times \left(\frac{1}{2|k_2+k_3||k_3+k_1|} + \frac{1}{2|k_1+k_2||k_3+k_1|} + \frac{1}{2|k_1+k_2||k_2+k_3|} + \frac{1}{|k_1+k_2||k_3+k_1|} + \frac{1}{|k_1||k_3+k_1|} + \frac{1}{|k_1||k_1+k_2|} \right).$$

By symmetry this is equal to

$$\frac{5}{2}\sum_{k_1,k_2,k_3}^{\text{nonres}} |u_{k_1}||v_{k_2}||w_{k_3}|\, \frac{\tilde{z}_{k_1+k_2+k_3}}{|k_2+k_3||k_3+k_1|} + 2\sum_{k_1,k_2,k_3}^{\text{nonres}} |u_{k_1}||v_{k_2}||w_{k_3}|\, \frac{\tilde{z}_{k_1+k_2+k_3}}{|k_1||k_1+k_2|} \le$$
$$\frac{5}{2}\|u\|\|v\|\|w\| \left(\sum_{k_1,k_2,k_3}^{\text{nonres}} \frac{\tilde{z}^2_{k_1+k_2+k_3}}{|k_2+k_3|^2|k_3+k_1|^2} \right)^{1/2} + 2\|u\|\|v\|\|w\| \left(\sum_{k_1,k_2,k_3}^{\text{nonres}} \frac{\tilde{z}^2_{k_1+k_2+k_3}}{|k_1|^2|k_1+k_2|^2} \right)^{1/2} \le$$

setting $k_2+k_3=l$, $k_3+k_1=j$, $k_1+k_2+k_3=k$ in the first sum, where $l,j,k \ne 0$,

$$\|u\|\|v\|\|w\| \left(\frac{5}{2} \left(\sum_l l^{-2} \sum_j j^{-2} \sum_k \tilde{z}_k^2 \right)^{1/2} + 2\left(\sum_{k_1} |k_1|^{-2} \sum_{k_2} |k_1+k_2|^{-2} \sum_{k_3} \tilde{z}^2_{k_1+k_2+k_3} \right)^{1/2} \right)$$
$$= \frac{9}{2}\left(\sum_i i^{-2} \right) \|u\|\|v\|\|w\|\|z\|_{\dot{H}^{-2}} = \frac{3\pi^2}{2}\|u\|\|v\|\|w\|\|z\|_{\dot{H}^{-2}}.$$

The proof in the case $s \ge 0$ is similar. We use the inequality that for $k_1 \cdot k_2 \cdot k_3 \ne 0$ $|k_1+k_2+k_3| \le 3|k_1||k_2||k_3|$. Then, setting $\tilde{u}_k = |k|^s|u_k|$, $\tilde{v}_k = |k|^s|v_k|$, $\tilde{w}_k = |k|^s|w_k|$, $\tilde{z}_k = |z_k|/|k|^{s+2}$, and following the above argument we have

$$|(B_3(u,v,w),z)| \le \sum_{k_1,k_2,k_3}^{\text{nonres}} \frac{\tilde{u}_{k_1}\tilde{v}_{k_2}\tilde{w}_{k_3}\, \tilde{z}_{k_1+k_2+k_3}|k_1+k_2+k_3|^2}{|k_1||k_1+k_2||k_2+k_3||k_3+k_1|} \frac{|k_1+k_2+k_3|^s}{|k_1|^s|k_2|^s|k_3|^s} \le$$
$$3^s \sum_{k_1,k_2,k_3}^{\text{nonres}} \frac{\tilde{u}_{k_1}\tilde{v}_{k_2}\tilde{w}_{k_3}\, \tilde{z}_{k_1+k_2+k_3}|k_1+k_2+k_3|^2}{|k_1||k_1+k_2||k_2+k_3||k_3+k_1|} \le 3^s\frac{3\pi^2}{2}\|u\|_{\dot{H}^s}\|v\|_{\dot{H}^s}\|w\|_{\dot{H}^s}\|z\|_{\dot{H}^{-s-2}},$$

which proves the lemma with $c_3(s) = 3^{(s+1)}\pi^2/2$. □

Remark 7.1. In fact, one can show that

$$\|B_3(u,v,w)\|_{\dot{H}^{2-4\eta}} \le c_3'(\eta)\|u\|_{\dot{H}^{-\eta}}\|v\|_{\dot{H}^{-\eta}}\|w\|_{\dot{H}^{-\eta}}, \qquad \eta < \frac{1}{4}. \tag{7.13}$$

Lemma 7.7. *Let $1 \ge s > 0$. Then the trilinear operator $B^{(n)}_{30}(v,v,v)$ defined in (6.15) for every t maps $\dot{H}^s \times \dot{H}^s \times \dot{H}^s$ into $\dot{H}^s$ and satisfies the estimate*

$$\|B^{(n)}_{30}(v,v,v)\|_{\dot{H}^s} \le \frac{\pi^2}{n^s}\|v\|^2_{\dot{H}^0}\|v\|_{\dot{H}^s}. \tag{7.14}$$

Proof. To estimate the norm of $B_{30}^{(n)}(v,v,v)$ we take $z \in \dot{H}^{-s}$ and consider

$$(B_{30}^{(n)}(v,v,v),z) = \sum_{k\in\mathbb{Z}_0} \sum_{k_1+k_2+k_3=k}^{\text{nonres}} \frac{e^{i3(k_1+k_2)(k_2+k_3)(k_3+k_1)t}}{k_1(k_1+k_2)(k_2+k_3)(k_3+k_1)} v_{k_1}\Pi_{-n}v_{k_2}\Pi_{-n}v_{k_3}z_k.$$

We set $\tilde{z}_k = |z_k|/|k|^s$ and using (7.12) obtain for $1 \geq s > 0$

$$|(B_{30}^{(n)}(v,v,v),z)| \leq \sum_{k\in\mathbb{Z}_0} \sum_{k_1+k_2+k_3=k}^{\text{nonres}} \frac{|k_1+k_2+k_3|^s}{|k_1||k_1+k_2||k_2+k_3||k_3+k_1|} |v_{k_1}||\Pi_{-n}v_{k_2}||\Pi_{-n}v_{k_3}|\tilde{z}_k \leq$$

$$\sum_{k\in\mathbb{Z}_0} \sum_{k_1+k_2+k_3=k}^{\text{nonres}} \Bigg(\frac{|v_{k_1}||k_2|^s|\Pi_{-n}v_{k_2}||\Pi_{-n}v_{k_3}|\tilde{z}_k}{|k_1|^{1-s}|k_1+k_2||k_2|^s|k_2+k_3||k_3+k_1|} +$$

$$\frac{|v_{k_1}||\Pi_{-n}v_{k_2}||k_3|^s|\Pi_{-n}v_{k_3}|\tilde{z}_k}{|k_1||k_1+k_2|^{1-s}|k_2+k_3||k_3+k_1||k_3|^s} + \frac{|v_{k_1}||k_2|^s|\Pi_{-n}v_{k_2}||\Pi_{-n}v_{k_3}|\tilde{z}_k}{|k_2|^s|k_1||k_1+k_2||k_2+k_3||k_3+k_1|^{1-s}} \Bigg) \leq$$

$$\frac{1}{n^s} \sum_{k_1,k_2,k_3}^{\text{nonres}} \Bigg(\frac{|v_{k_1}|\,|k_2|^s\,|\Pi_{-n}v_{k_2}|\,|\Pi_{-n}v_{k_3}|}{|k_1|^{1-s}\,|k_1+k_2|\,|k_2+k_3|\,|k_3+k_1|} +$$

$$\frac{|v_{k_1}|\,|\Pi_{-n}v_{k_2}|\,|k_3|^s\,|\Pi_{-n}v_{k_3}|}{|k_1|\,|k_1+k_2|^{1-s}\,|k_2+k_3|\,|k_3+k_1|} + \frac{|v_{k_1}|\,|k_2|^s\,|\Pi_{-n}v_{k_2}|\,|\Pi_{-n}v_{k_3}|}{|k_1|\,|k_1+k_2|\,|k_2+k_3|\,|k_3+k_1|^{1-s}} \Bigg) \tilde{z}_{k_1+k_2+k_3} \leq$$

$$\frac{1}{n^s}\|v\|_{\dot{H}^0}^2\|v\|_{\dot{H}^s} \Bigg[\sum_{k_1,k_2,k_3}^{\text{nonres}} \frac{\tilde{z}_{k_1+k_2+k_3}^2}{|k_1|^{2-2s}\,|k_1+k_2|^2\,|k_2+k_3|^2\,|k_3+k_1|^2} +$$

$$\frac{\tilde{z}_{k_1+k_2+k_3}^2}{|k_1|^2\,|k_1+k_2|^{2-2s}\,|k_2+k_3|^2\,|k_3+k_1|^2} + \frac{\tilde{z}_{k_1+k_2+k_3}^2}{|k_1|^2\,|k_1+k_2|^2\,|k_2+k_3|^2\,|k_3+k_1|^{2-2s}} \Bigg]^{1/2} \leq$$

$$\frac{3}{n^s}\|v\|_{\dot{H}^0}^2\|v\|_{\dot{H}^s} \Bigg[\sum_{k_1,k_2,k_3}^{\text{nonres}} \frac{\tilde{z}_{k_1+k_2+k_3}^2}{|k_1+k_2|^2|k_2+k_3|^2} \Bigg]^{1/2} = \frac{\pi^2}{n^s}\|v\|_{\dot{H}^0}^2\|v\|_{\dot{H}^s}\|z\|_{\dot{H}^{-s}}.$$

□

For $s \leq 0$ we have the following estimate.

Lemma 7.8. *Let $s \leq 0$. Then the trilinear operator $B_{30}^{(n)}(v,v,v)$ satisfies the estimate*

$$\|B_{30}^{(n)}(v,v,v)\|_{\dot{H}^s} \leq \frac{C(p,\alpha)}{n^{2\alpha}}\|v\|_{\dot{H}^s}^3, \tag{7.15}$$

where $p = -s \geq 0$, $\alpha > 0$ and $p + \alpha < 5/6$.

Proof. If $s = -p \leq 0$ we set $\tilde{v}_k = |v_k||k|^s$, $\tilde{z}_k = |z_k|/|k|^s$ so that $\|\tilde{v}\|_{\dot{H}^0} = \|v\|_{\dot{H}^s}$, $\|\tilde{z}\|_{\dot{H}^0} = \|z\|_{\dot{H}^{-s}}$ and obtain

$$|(B_{30}^{(n)}(v,v,v),z)| \leq \sum_{k\in\mathbb{Z}_0} \sum_{k_1+k_2+k_3=k}^{\text{nonres}} \frac{|k_1|^p\tilde{v}_{k_1}|k_2|^p\Pi_{-n}\tilde{v}_{k_2}|k_3|^p\Pi_{-n}\tilde{v}_{k_3}\tilde{z}_k}{|k_1||k_1+k_2||k_2+k_3||k_3+k_1||k_1+k_2+k_3|^p} \leq$$

$$\frac{1}{n^{2\alpha}} \sum_{k\in\mathbb{Z}_0} \sum_{k_1+k_2+k_3=k}^{\text{nonres}} K(k_1,k_2,k_3)\,\tilde{v}_{k_1}\Pi_{-n}\tilde{v}_{k_2}\Pi_{-n}\tilde{v}_{k_3}\tilde{z}_k,$$

where

$$K(k_1,k_2,k_3) = \frac{|k_2|^{p+\alpha}\,|k_3|^{p+\alpha}}{|k_1|^{1-p}\,|k_1+k_2|\,|k_2+k_3|\,|k_3+k_1|\,|k_1+k_2+k_3|^p}. \tag{7.16}$$

Now we include condition $|k|>0$ into definition of nonresonant summation and write

$$\left|\left(B_{30}^{(n)}(v,v,v),z\right)\right| \leq \frac{1}{n^{2\alpha}} \sum_{k_1,k_2,k_3}^{\text{nonres}} K\,|\tilde{v}_{k_1}|\,|\tilde{v}_{k_2}|\,|\tilde{v}_{k_3}|\,\tilde{z}_k = \tag{7.17}$$

$$\frac{1}{n^{2\alpha}} \sum_{k_1,k_2,k_3}^{\text{nonres}} (|\tilde{v}_{k_1}|\,|\tilde{v}_{k_2}|\,|\tilde{v}_{k_3}|)^{1/3} (|\tilde{v}_{k_2}|\,|\tilde{v}_{k_3}|\,\tilde{z}_k)^{1/3} (|\tilde{v}_{k_1}|\,|\tilde{v}_{k_3}|\,\tilde{z}_k)^{1/3} (|\tilde{v}_{k_1}|\,|\tilde{v}_{k_2}|\,\tilde{z}_k)^{1/3} K \leq$$

$$\leq \frac{1}{n^{2\alpha}} \left(\sum_{k_1,k_2,k_3}^{\text{nonres}} (|\tilde{v}_{k_1}|\,|\tilde{v}_{k_2}|\,|\tilde{v}_{k_3}|)^2\right)^{1/6} \left(\sum_{k_1,k_2,k_3}^{\text{nonres}} (|\tilde{v}_{k_2}|\,|\tilde{v}_{k_3}|\,\tilde{z}_k)^2\right)^{1/6} \times$$

$$\left(\sum_{k_1,k_2,k_3}^{\text{nonres}} (|\tilde{v}_{k_1}|\,|\tilde{v}_{k_3}|\,\tilde{z}_k)^2\right)^{1/6} \left(\sum_{k_1,k_2,k_3}^{\text{nonres}} (|\tilde{v}_{k_1}|\,|\tilde{v}_{k_2}|\,\tilde{z}_k)^2\right)^{1/6} \left(\sum_{k_1,k_2,k_3}^{\text{nonres}} K^3\right)^{1/3} =$$

$$\frac{1}{n^{2\alpha}} \|\tilde{v}\|^3_{\dot{H}^0} \|\tilde{z}\|_{\dot{H}^0} \left(\sum_{k_1,k_2,k_3}^{\text{nonres}} K^3\right)^{1/3} = \frac{1}{n^{2\alpha}} \|v\|^3_{\dot{H}^s} \|z\|_{\dot{H}^{-s}} \left(\sum_{k_1,k_2,k_3}^{\text{nonres}} K^3\right)^{1/3},$$

where $\left(\sum_{k_1,k_2,k_3}^{\text{nonres}} K^3\right)^{1/3} =: C(p,\alpha) < \infty$ by Lemma 7.9 (where $\gamma=\delta := p+\alpha$). □

Lemma 7.9. *Let $0 \leq p \leq 1$ and let $K(k_1,k_2,k_3)$ be defined as follows*

$$K\left(k_1,k_2,k_3\right) = \frac{|k_2|^\gamma |k_3|^\delta}{|k_1|^{1-p}|k_1+k_2||k_2+k_3||k_3+k_1||k_1+k_2+k_3|^p}. \tag{7.18}$$

If

$$\gamma+\delta < \frac{5}{3}, \tag{7.19}$$

then

$$\sum_{k_1,k_2,k_3}^{\text{nonres}} K(k_1,k_2,k_3)^3 \leq C(\gamma,\delta,p) < \infty.$$

Proof. By (7.18)

$$K(k_1,k_2,k_3)^3 = \frac{|k_2|^{3\gamma}|k_3|^{3\delta}}{|k_1|^{3-3p}|k_1+k_2|^3|k_2+k_3\|^3|k_3+k_1|^3|k_1+k_2+k_3|^{3p}}.$$

We denote

$$l_1 = k_1+k_2,\ l_2 = k_2+k_3,\ l_3 = k_1+k_3,\ l_0 = k = k_1+k_2+k_3.$$

Obviously,

$$k_1 = \frac{1}{2}\left(l_3+l_1-l_2\right), k_3 = \frac{1}{2}\left(l_3-l_1+l_2\right), k_2 = \frac{1}{2}\left(l_2-l_3+l_1\right), k = \frac{1}{2}\left(l_2+l_3+l_1\right)$$

and setting $k_1 = l_4$, $k = k_1+k_2+k_3 = l_0$ we have

$$K\left(k_1,k_2,k_3\right)^3 = \frac{|k_2|^{3\gamma}|k_3|^{3\delta}}{\left(|l_4||l_1||l_2||l_3|\right)^{3-3p}\left(|l_0||l_1||l_2||l_3|\right)^{3p}}$$

and therefore

$$\sum_{k_1,k_2,k_3}^{\text{nonres}} K(k_1,k_2,k_3)^3 \leq \left(\sum_{k_1,k_2,k_3}^{\text{nonres}} \frac{|k_2|^{3\gamma}|k_3|^{3\delta}}{(|l_4||l_1||l_2||l_3|)^3}\right)^{1-p} \left(\sum_{k_1,k_2,k_3}^{\text{nonres}} \frac{|k_2|^{3\gamma}|k_3|^{3\delta}}{(|l_0||l_1||l_2||l_3|)^3}\right)^p. \tag{7.20}$$

We have four linear functions $l_1, l_2, l_3, k = l_0$ defined on the (k_1, k_2, k_3) space, any three of the four functions are linearly independent. Therefore

$$|k_2| + |k_3| \leq C_1 \sum_{i \neq j} |l_i|, \quad \text{for} \quad j = 0, 1, 2, 3.$$

Similarly, four linear functions $l_1, l_2, l_3, k_1 = l_4$ are defined on the (k_1, k_2, k_3) space, any three of them are linearly independent. Therefore

$$|k_2| + |k_3| \leq C_2 \sum_{i \neq j} |l_i|, \quad \text{for} \quad j = 1, 2, 3, 4.$$

Hence

$$|k_2| + |k_3| \leq C_3 \left(\prod_{j=0}^{3} \sum_{i \neq j} |l_i| \right)^{1/4}, \quad i = 0, 1, 2, 3.$$

$$|k_2| + |k_3| \leq C_3 \left(\prod_{j=0}^{3} \sum_{i \neq j} |l_i| \right)^{1/4}, \quad i = 1, 2, 3, 4.$$

Using inequalities of the type $l_0 l_1 l_2 l_3 \leq 1/2(l_0^2 l_1^2 + l_2^2 l_3^2) \leq 1/4(l_0 l_1^3 + l_0^3 l_1 + l_2 l_3^3 + l_2^3 l_3)$ we obtain

$$|k_2| + |k_3| \leq C_4 \sum_{j=0}^{3} \sum_{i \neq j} (|l_j||l_i|^3)^{1/4}, \quad |k_2| + |k_3| \leq C_4 \sum_{j=1}^{4} \sum_{i \neq j} (|l_j||l_i|^3)^{1/4}$$

and

$$\sum_{k_1,k_2,k_3}^{\text{nonres}} \frac{|k_2|^{3\gamma}|k_3|^{3\delta}}{(|l_4||l_1||l_2||l_3|)^3} \leq C_5 \sum_{j=1}^{4} \sum_{i \neq j} \sum_{k_1,k_2,k_3}^{\text{nonres}} \frac{(|l_i||l_j|^3)^{3(\gamma+\delta)/4}}{(|l_4||l_1||l_2||l_3|)^3}.$$

Each term in the above sum can be estimated quite similarly, we take, for example, $j = 2$, $i = 1$ (all remaining combinations are obtained by an obvious permutation of indices). Expressing below l_2 as the linear combination of l_1, l_3 and l_4 we get

$$\sum_{k_1,k_2,k_3}^{\text{nonres}} \frac{(|l_1||l_2|^3)^{3(\gamma+\delta)/4}}{(|l_4||l_1||l_2||l_3|)^3} = \sum_{k_1,k_2,k_3}^{\text{nonres}} \frac{|l_1|^{3(\gamma+\delta)/4}|l_2|^{9(\gamma+\delta)/4}}{(|l_4||l_1||l_2||l_3|)^3} = \sum_{k_1,k_2,k_3}^{\text{nonres}} \frac{|l_1|^{3(\gamma+\delta)/4}|l_2|^{9(\gamma+\delta)/4-3}}{|l_4|^3|l_1|^3|l_3|^3} \leq$$
$$C_5' \sum_{k_1,k_2,k_3}^{\text{nonres}} \frac{|l_4|^{3(\gamma+\delta)/4+9(\gamma+\delta)/4-3} + |l_1|^{3(\gamma+\delta)/4+9(\gamma+\delta)/4-3} + |l_3|^{3(\gamma+\delta)/4+9(\gamma+\delta)/4-3}}{|l_4|^3|l_1|^3|l_3|^3}.$$

The series converges if $3(\gamma+\delta)/4 + 9(\gamma+\delta)/4 - 3 < 2$, which is condition (7.19). Since the second factor in (7.20) is treated in exactly the same way, the proof is complete. $\square$

Lemma 7.10. *If $0 < s \leq 1$, then*

$$\|B_{30}^{(n)}(u,u,v)\|_{\dot{H}^s} + \|B_{30}^{(n)}(u,v,u)\|_{\dot{H}^s} + \|B_{30}^{(n)}(v,u,u)\|_{\dot{H}^s} \leq \frac{C}{n^s}\|u\|_{\dot{H}^0}^2\|v\|_{\dot{H}^s}. \tag{7.21}$$

If $s \leq 0$ and $p = -s \leq 1$, $\alpha > 0$, $p + 2\alpha < 5/3$, then

$$\|B_{30}^{(n)}(u,u,v)\|_{\dot{H}^s} + \|B_{30}^{(n)}(u,v,u)\|_{\dot{H}^s} + \|B_{30}^{(n)}(v,u,u)\|_{\dot{H}^s} \leq \frac{C(p,\alpha)}{n^{2\alpha}}\|u\|_{\dot{H}^0}^2\|v\|_{\dot{H}^s}. \tag{7.22}$$

Proof. We consider the second case. If $-s = p \le 1$ we set $\tilde{v}_k = |v_k||k|^s$, and for $z \in \dot{H}^{-s}$ we set $\tilde{z}_k = |z_k||k|^{-s} = |z_k||k|^p$. Note that $B_{30}^{(n)}(u,u,v) = B_{30}^{(n)}(u,v,u)$, so we consider only $B_{30}^{(n)}(u,u,v)$ and $B_{30}^{(n)}(v,u,u)$. Using duality we write

$$|(B_{30}^{(n)}(u,u,v),z)| \le$$
$$\sum_{k\in\mathbb{Z}_0}\sum_{k_1+k_2+k_3=k}^{\text{nonres}} \frac{|u_{k_1}||\Pi_{-n}u_{k_2}||k_3|^p|\Pi_{-n}\tilde{v}_{k_3}|\tilde{z}_k}{|k_1||k_1+k_2||k_2+k_3||k_3+k_1||k_1+k_2+k_3|^p} \le$$
$$\frac{1}{n^{2\alpha}}\sum_{k\in\mathbb{Z}_0}\sum_{k_1+k_2+k_3=k}^{\text{nonres}} K_2\,|u_{k_1}||\Pi_{-n}u_{k_2}||\Pi_{-n}\tilde{v}_{k_3}|\tilde{z}_k =$$
$$\frac{1}{n^{2\alpha}}\sum_{k_1,k_2,k_3}^{\text{nonres}} K_2\,|u_{k_1}|\,|\Pi_{-n}u_{k_2}|\,|\Pi_{-n}\tilde{v}_{k_3}|\,\tilde{z}_{k_1+k_2+k_3},$$

where

$$K_2(k_1,k_2,k_3) = \frac{|k_2|^\alpha|k_3|^{p+\alpha}}{|k_1||k_1+k_2||k_2+k_3||k_3+k_1||k_1+k_2+k_3|^p}.$$

Now we write similarly to (7.17)

$$|(B_{30}^{(n)}(u,u,v),z)| \le \frac{1}{n^{2\alpha}}\sum_{k_1,k_2,k_3}^{\text{nonres}} K_2|u_{k_1}||u_{k_2}||\tilde{v}_{k_3}|\tilde{z}_k \le$$
$$\frac{1}{n^{2\alpha}}\|u\|_{\dot{H}^0}^2\|v\|_{\dot{H}^s}\|z\|_{\dot{H}^{-s}}\left(\sum_{k_1,k_2,k_3}^{\text{nonres}} K_2^3\right)^{1/3}$$

Since $0 \le p \le 1$, $|k_1| \ge |k_1|^{1-p}$ and we can apply Lemma 7.9 with $\gamma = \alpha$ and $\delta = p+\alpha$, which gives that

$$\left(\sum_{k_1,k_2,k_3}^{\text{nonres}} K_2^3\right)^{1/3} \le C'(p,\alpha) < \infty,$$

provided that $p + 2\alpha < 5/3$.

Similarly,

$$|(B_{30,n}(v,u,u),z)| \le$$
$$\frac{1}{n^{2\alpha}}\sum_{k\in\mathbb{Z}_0}\sum_{k_1+k_2+k_3=k}^{\text{nonres}} \frac{|k_1|^p|\tilde{v}_{k_1}||k_2|^\alpha|\Pi_{-n}u_{k_2}||k_3|^\alpha|\Pi_{-n}u_{k_3}|\tilde{z}_k}{|k_1||k_1+k_2||k_2+k_3||k_3+k_1||k_1+k_2+k_3|^p} \le$$
$$\frac{1}{n^{2\alpha}}\|u\|_{\dot{H}^0}^2\|v\|_{\dot{H}^s}\|z\|_{\dot{H}^{-s}}\left(\sum_{k_1,k_2,k_3}^{\text{nonres}} K_3^3\right)^{1/3},$$

where

$$K_3(k_1,k_2,k_3) = \frac{|k_2|^\alpha\,|k_3|^\alpha}{|k_1|^{1-p}\,|k_1+k_2|\,|k_2+k_3|\,|k_3+k_1|\,|k_1+k_2+k_3|^p}.$$

By Lemma 7.9 the series converges if $2\alpha < 5/3$, and this condition follows from the previous one since $p \ge 0$. The first case (7.21) is simpler and is treated similarly to Lemma 7.7. $\square$

Lemma 7.11. *Let $s \ge 0$ and $\varepsilon \in (0, \frac{1}{2})$. Then the multi-linear operator B_4 defined in (3.26) maps $(\dot{H}^s)^4$ into $\dot{H}^{s+\varepsilon}$ and satisfies the estimate*

$$\|B_4(u,v,w,\varphi)\|_{\dot{H}^{s+\varepsilon}} \le c_4(s,\varepsilon)\|u\|_{\dot{H}^s}\|v\|_{\dot{H}^s}\|w\|_{\dot{H}^s}\|\varphi\|_{\dot{H}^s}. \tag{7.23}$$

Proof. Since $B_4 = \frac{1}{2}B_4^1 + B_4^2$ (see (3.27)), it suffices to estimate B_4^1 and B_4^2. We first consider the case $s = 0$. Setting $\tilde{z}_k = |z_k|/|k|^\varepsilon$, $\varepsilon < 1/2$ and using the inequality

$$|k_1 + k_2 + k_3 + k_4|^\varepsilon \le 2^{-\varepsilon}(|k_1 + k_2|^\varepsilon + |k_1 + k_3 + k_4|^\varepsilon + |k_2 + k_3 + k_4|^\varepsilon),$$

we have

$$|(B_4^1(u, v, w, \varphi), z)| \le \sum_{k_1,k_2,k_3,k_4}^{\text{nonres}} \frac{|u_{k_1}||v_{k_2}||w_{k_3}||\varphi_{k_4}|\, \tilde{z}_{k_1+k_2+k_3+k_4}\, |k_1 + k_2 + k_3 + k_4|^\varepsilon}{|k_1 + k_2||k_1 + k_3 + k_4||k_2 + k_3 + k_4|} \le$$

$$2^{-\varepsilon} \sum_{k_1,k_2,k_3,k_4}^{\text{nonres}} |u_{k_1}||v_{k_2}||w_{k_3}||\varphi_{k_4}|\, \tilde{z}_{k_1+k_2+k_3+k_4} \biggl(\frac{1}{|k_1 + k_2|^{1-\varepsilon}|k_1 + k_3 + k_4||k_2 + k_3 + k_4|} +$$

$$+ \frac{1}{|k_1 + k_2||k_1 + k_3 + k_4|^{1-\varepsilon}|k_2 + k_3 + k_4|} + \frac{1}{|k_1 + k_2||k_1 + k_3 + k_4||k_2 + k_3 + k_4|^{1-\varepsilon}} \biggr).$$

The three terms so obtained satisfy the same bound, and it suffices to consider any of them, say, the first:

$$\sum_{k_1,k_2,k_3,k_4}^{\text{nonres}} \frac{|u_{k_1}||v_{k_2}||w_{k_3}||\varphi_{k_4}|\, \tilde{z}_{k_1+k_2+k_3+k_4}}{|k_1 + k_2|^{1-\varepsilon}|k_1 + k_3 + k_4||k_2 + k_3 + k_4|} \le$$

$$\|u\|\|v\|\|w\|\|\varphi\| \biggl(\sum_{k_1,k_2,k_3,k_4}^{\text{nonres}} \frac{\tilde{z}^2_{k_1+k_2+k_3+k_4}}{|k_1 + k_2|^{2-2\varepsilon}|k_1 + k_3 + k_4|^2|k_2 + k_3 + k_4|^2} \biggr)^{1/2} \le$$

$$\|u\|\|v\|\|w\|\|\varphi\| \biggl(\sum_{l_1} |l_1|^{-2+2\varepsilon} \sum_{l_2} (l_2 + l_1)^{-2} \sum_{l_3} l_3^{-2} \sum_{l_0} \tilde{z}^2_{l_0} \biggr)^{1/2} =$$

$$c(1 - \varepsilon)\frac{\pi^2}{3}\|u\|\|v\|\|w\|\|\varphi\|\|z\|_{\dot{H}^{-\varepsilon}},$$

where $c(1 - \varepsilon)$ is as in (7.4) and where we set

$$\begin{array}{ccccccccc} k_1 & + & k_2 & & & & & = & l_1 \\ & & & & k_3 & & & = & l_2 \\ & & k_2 & + & k_3 & + & k_4 & = & l_3 \\ k_1 & + & k_2 & + & k_3 & + & k_4 & = & l_0, \end{array} \qquad \text{where} \qquad l_1, l_2, l_3, l_0 \in \mathbb{Z}_0.$$

Hence,

$$|(B_4^1(u, v, w, \varphi), z)| \le c(1 - \varepsilon)2^{-\varepsilon}\pi^2\|u\|\|v\|\|w\|\|\varphi\|\|z\|_{\dot{H}^{-\varepsilon}}\,. \tag{7.24}$$

It remains to estimate B_4^2. Since $|k_3 + k_4| \le |k_1 + k_3 + k_4| + |k_1|$, we have

$$|(B_4^2(u, v, w, \varphi), z)| \le \sum_{k_1,k_2,k_3,k_4}^{\text{nonres}} \frac{|k_3 + k_4||u_{k_1}||v_{k_2}||w_{k_3}||\varphi_{k_4}|\, \tilde{z}_{k_1+k_2+k_3+k_4}\, |k_1 + k_2 + k_3 + k_4|^\varepsilon}{|k_1||k_1 + k_2||k_1 + k_3 + k_4||k_2 + k_3 + k_4|} \le$$

$$\sum_{k_1,k_2,k_3,k_4}^{\text{nonres}} \frac{|u_{k_1}||v_{k_2}||w_{k_3}||\varphi_{k_4}|\, \tilde{z}_{k_1+k_2+k_3+k_4}|k_1 + k_2 + k_3 + k_4|^\varepsilon}{|k_1||k_1 + k_2||k_2 + k_3 + k_4|} +$$

$$\sum_{k_1,k_2,k_3,k_4}^{\text{nonres}} \frac{|u_{k_1}||v_{k_2}||w_{k_3}||\varphi_{k_4}|\, \tilde{z}_{k_1+k_2+k_3+k_4}|k_1 + k_2 + k_3 + k_4|^\varepsilon}{|k_1 + k_2||k_1 + k_3 + k_4||k_2 + k_3 + k_4|}.$$

As shown above, the second sum is bounded by the right-hand side in (7.24). We split the first sum using the inequality $|k_1 + k_2 + k_3 + k_4|^\varepsilon \le |k_1|^\varepsilon + |k_2 + k_3 + k_4|^\varepsilon$. Then the first

sum is bounded by

$$\sum_{k_1,k_2,k_3,k_4}^{\text{nonres}} \frac{|u_{k_1}||v_{k_2}||w_{k_3}||\varphi_{k_4}|\,\tilde z_{k_1+k_2+k_3+k_4}}{|k_1|^{1-\varepsilon}|k_1+k_2||k_2+k_3+k_4|} + \sum_{k_1,k_2,k_3,k_4}^{\text{nonres}} \frac{|u_{k_1}||v_{k_2}||w_{k_3}||\varphi_{k_4}|\,\tilde z_{k_1+k_2+k_3+k_4}}{|k_1||k_1+k_2||k_2+k_3+k_4|^{1-\varepsilon}}.$$

Both terms are treated in the same way and satisfy the same bound. Let us consider the second one. Setting $k_1+k_2+k_3+k_4 = l_0$, so that $k_4 = l_0 - k_1 - k_2 - k_3$ and $k_2+k_3+k_4 = l_0 - k_1$ we have

$$\sum_{k_1,k_2,k_3,k_4}^{\text{nonres}} \frac{|u_{k_1}||v_{k_2}||w_{k_3}||\varphi_{k_4}|\,\tilde z_{k_1+k_2+k_3+k_4}}{|k_1||k_1+k_2||k_2+k_3+k_4|^{1-\varepsilon}} = \sum_{k_1,k_2,k_3,l_0}^{\text{nonres}} \frac{|u_{k_1}||v_{k_2}||w_{k_3}||\varphi_{l_0-k_1-k_2-k_3}|\,\tilde z_l}{|k_1||k_1+k_2||l_0-k_1|^{1-\varepsilon}} =$$

$$\sum_{k_1}\sum_{k_2}\frac{|u_{k_1}||v_{k_2}|}{|k_1||k_1+k_2|}\sum_{l_0\neq k_1}|l_0-k_1|^{-1+\varepsilon}\,\tilde z_{l_0}\sum_{k_3}|w_{k_3}||\varphi_{l_0-k_1-k_2-k_3}| \le$$

$$\sum_{k_1,k_2}\frac{|u_{k_1}||v_{k_2}|}{|k_1||k_1+k_2|}c(1-\varepsilon)\|z\|_{\dot H^{-\varepsilon}}\|w\|\|\varphi\| \le$$

$$\left(\sum_{k_1}\sum_{k_2}\frac{1}{|k_1|^2|k_1+k_2|^2}\right)^{1/2} c(1-\varepsilon)\|u\|\|v\|\|w\|\|\varphi\|\|z\|_{\dot H^{-\varepsilon}} =$$

$$c(1-\varepsilon)\frac{\pi^2}{3}\|u\|\|v\|\|w\|\|\varphi\|\|z\|_{\dot H^{-\varepsilon}}.$$

Hence,

$$|(B_4^2(u,v,w,\varphi),z)| \le c(1-\varepsilon)\pi^2(2^{-\varepsilon}+2/3)\|u\|\|v\|\|w\|\|\varphi\|\|z\|_{\dot H^{-\varepsilon}},$$

and B_4 satisfies the estimate

$$\|B_4(u,v,w,\varphi)\|_{\dot H^{s+\varepsilon}} \le c_4(s,\varepsilon)\|u\|_{\dot H^s}\|v\|_{\dot H^s}\|w\|_{\dot H^s}\|\varphi\|_{\dot H^s}, \quad \varepsilon < \frac{1}{2} \tag{7.25}$$

with $s = 0$ and $c_4(0,\varepsilon) = c(1-\varepsilon)\pi^2(3\cdot 2^{-\varepsilon-1}+2/3)$. Finally, the general case $s \ge 0$ is treated as in Lemma 7.6 by using the inequality

$$\frac{|k_1+k_2+k_3+k_4|}{|k_1||k_2||k_3||k_4|} \le 4,$$

which proves (7.25) with $c_4(s,\varepsilon) = 4^s c_4(0,\varepsilon)$. □

Lemma 7.12. *Let $-3/2 < s < 1/2$. Then the nonlinear operator B_{40}^1 given by (6.17) satisfies the estimate*

$$\|B_{40}^1(u,u,u,v)\|_{\dot H^s} + \|B_{40}^1(v,u,u,u)\|_{\dot H^s} + \|B_{40}^1(u,v,u,u)\|_{\dot H^s} + \|B_{40}^1(u,u,v,u)\|_{\dot H^s} \le c_8(s)\|u\|_{\dot H^0}^3\|v\|_{\dot H^s}. \tag{7.26}$$

If $1/2 \le s$ and $\theta_0 > s - 1/2$, then

$$\|B_{40}^1(u,u,u,v)\|_{\dot H^s} + \|B_{40}^1(v,u,u,u)\|_{\dot H^s} + \|B_{40}^1(u,v,u,u)\|_{\dot H^s} + \|B_{40}^1(u,u,v,u)\|_{\dot H^s} \le c_8(s,\theta_0)\|u\|_{\dot H^{\theta_0}}^3\|v\|_{\dot H^s}. \tag{7.27}$$

Proof. We observe that in the proof below we do not take advantage of the operators Π_{-n} in (6.17) and replace them with identity operators.

We first consider a more difficult case of negative s: $-3/2 < s \le 0$. We set $p = -s$. Using duality, we estimate

$$|(B^1_{40}(v,u,u,u),z)| \le$$
$$\sum_{k_1,k_2,k_3,k_4}^{\text{nonres}} \frac{|k_1|^p |\tilde v_{k_1}||u_{k_2}||u_{k_3}||u_{k_4}|\tilde z_{k_1+k_2+k_3+k_4}}{|k_1+k_2||k_1+k_3+k_4||k_2+k_3+k_4||k_1+k_2+k_3+k_4|^p} \le$$
$$\|\tilde z\|_{\dot H^0}\|u\|^2_{\dot H^0}\|v\|_{\dot H^s}\times$$
$$\left(\sum_{k_1,k_2,k_3,k_4}^{\text{nonres}} \frac{|k_1|^{2p}|u_{k_3}|^2}{|k_1+k_2|^2|k_1+k_3+k_4|^2|k_2+k_3+k_4|^2|k_1+k_2+k_3+k_4|^{2p}}\right)^{1/2} =$$
$$\|\tilde z\|_{\dot H^0}\|u\|^2_{\dot H^0}\|v\|_{\dot H^s}\left(\sum_{k_1,k_2,k_3,l_0}^{\text{nonres}} \frac{|k_1|^{2p}|u_{k_3}|^2}{|k_1+k_2|^2|l_0-k_1|^2|l_0-k_2|^2|l_0|^{2p}}\right)^{1/2} =$$
$$\|\tilde z\|_{\dot H^0}\|u\|^3_{\dot H^0}\|v\|_{\dot H^s}\left(\sum_{k_1,k_2,l_0}^{\text{nonres}} K_{41}\right)^{1/2},$$

where $\tilde z_k = |z_k||k|^{-s}$, $\tilde v_k = |v_k||k|^s$, $l_0 = k_1+k_2+k_3+k_4$ and where

$$K_{41}(k_1,k_2,l_0) = \frac{|k_1|^{2p}}{|k_1+k_2|^2|l_0-k_1|^2|l_0-k_2|^2|l_0|^{2p}}.$$

Since any three of the four linear functionals l_j: $l_0 = k_1+k_2+k_3+k_4$, $l_1 = k_1+k_2$, $l_2 = l_0-k_1$, $l_3 = l_0-k_2$ are linearly independent over the 3-dimensional space of vectors (k_1,k_2,l_0), similarly to Lemma 7.9 we have

$$\sum_{k_1,k_2,l_0}^{\text{nonres}} K_{41} \le C\sum_{i\ne j}\sum_{k_1,k_2,l_0}^{\text{nonres}} \frac{(l_j^3 l_i)^{p/2}}{|l_1|^2|l_2|^2|l_3|^2|l_0|^{2p}}.$$

All the terms are estimated similarly, we take two typical examples:

$$\sum_{k_1,k_2,l_0}^{\text{nonres}} \frac{(l_0^3 l_1)^{p/2}}{|l_1|^2|l_2|^2|l_3|^2|l_0|^{2p}} = \sum_{l_3}^{\text{nonres}}\sum_{l_2}^{\text{nonres}} \frac{1}{|l_2|^2|l_3|^2}\sum_{l_0}^{\text{nonres}} \frac{1}{|l_1|^{2-p/2}|l_0|^{p/2}} \le$$
$$\sum_{l_3}^{\text{nonres}}\sum_{l_2}^{\text{nonres}} \frac{1}{|l_2|^2|l_3|^2}\left(\sum_{l_0}^{\text{nonres}} \frac{1}{|l_1|^2}\right)^{1-p/4}\left(\sum_{l_1}^{\text{nonres}} \frac{1}{|l_0|^2}\right)^{p/4},$$

which is a finite constant for $p \le 4$. Similarly,

$$\sum_{k_1,k_2,l_0}^{\text{nonres}} \frac{(l_3^3 l_1)^{p/2}}{|l_1|^2|l_2|^2|l_3|^2|l_0|^{2p}} = \sum_{l_1}^{\text{nonres}}\sum_{l_2}^{\text{nonres}} \frac{1}{|l_1|^{2-p/2}|l_2|^2}\sum_{l_3}^{\text{nonres}} \frac{1}{|l_3|^{2-3p/2}|l_0|^{2p}}.$$

First, we take $p < 4/3$. Then $2-3p/2 > 0$ and

$$\sum_{l_3}^{\text{nonres}} \frac{1}{|l_3|^{2-3p/2}|l_0|^{2p}} \le \left(\sum_{l_3}^{\text{nonres}} \frac{1}{|l_3|^{(2-3p/2)/(1-q)}}\right)^{1-q}\left(\sum_{l_3}^{\text{nonres}} \frac{1}{|l_0|^{2p/q}}\right)^{q}$$

with $2p/q > 1$, $0 \le q \le 1$, $2-3p/2 > 1-q$. Since $p < 4/3$ such a q exists and the series converges.

If $p \ge 4/3$ and $p < 3/2$, then we express l_3 as the unique linear combination of l_0, l_1, l_2 (namely, $l_3 = 2l_0 - l_1 - l_2$) and obtain

$$\sum_{k_1,k_2,l_0}^{\text{nonres}} \frac{(l_3^3 l_1)^{p/2}}{|l_1|^2|l_2|^2|l_3|^2|l_0|^{2p}} = \sum_{l_0,l_1,l_2}^{\text{nonres}} \frac{|l_3|^{3p/2-2}}{|l_1|^{2-p/2}|l_2|^2|l_0|^{2p}} \le C \sum_{l_0,l_1,l_2}^{\text{nonres}} \frac{|l_1|^{3p/2-2}+|l_2|^{3p/2-2}+|l_0|^{3p/2-2}}{|l_1|^{2-p/2}|l_2|^2|l_0|^{2p}}.$$

The series converges if $2 - p/2 + 2 - 3p/2 > 1$, $-3p/2 + 4 > 1$, $2 + p/2 > 1$, that is, if $3 > 2p$. The three remaining terms in (7.26) are treated in the same way. The proof of (7.26) for the case of negative s is complete.

For the case of positive s we observe that Lemma 7.11 gives the estimate

$$\|B^1_{40}(u,v,w,\varphi)\|_{\dot H^s} \le c_4(0,s)\|u\|_{\dot H^0}\|v\|_{\dot H^0}\|w\|_{\dot H^0}\|\varphi\|_{\dot H^0}, \quad 0 \le s < 1/2,$$

which implies (7.26) for the remaining interval $0 \le s < 1/2$.

For $s \ge 1/2$ we again use Lemma 7.11 to see that

$$\|B^1_{40}(u,v,w,\varphi)\|_{\dot H^s} \le c_4(\theta_0, s-\theta_0)\|u\|_{\dot H^{\theta_0}}\|v\|_{\dot H^{\theta_0}}\|w\|_{\dot H^{\theta_0}}\|\varphi\|_{\dot H^{\theta_0}}, \quad s-\theta_0 < 1/2,$$

which gives (7.27). □

Lemma 7.13. *Let $-3/2 < s < 1/2$. Then the nonlinear operator B^2_{40} given by (6.18) satisfies the estimate*

$$\begin{aligned}\|B^2_{40}(u,u,u,v)\|_{\dot H^s} + \|B^2_{40}(v,u,u,u)\|_{\dot H^s}+ \\ \|B^2_{40}(u,v,u,u)\|_{\dot H^s}+\|B^2_{40}(u,u,v,u)\|_{\dot H^s} \le c_8(s)\|u\|^3_{\dot H^0}\|v\|_{\dot H^s}.\end{aligned} \tag{7.28}$$

If $1/2 \le s$ and $\theta_0 > s - 1/2$, then

$$\begin{aligned}\|B^2_{40}(u,u,u,v)\|_{\dot H^s} + \|B^2_{40}(v,u,u,u)\|_{\dot H^s}+ \\ \|B^2_{40}(u,v,u,u)\|_{\dot H^s}+\|B^2_{40}(u,u,v,u)\|_{\dot H^s} \le c_8(s,\theta_0)\|u\|^3_{\dot H^{\theta_0}}\|v\|_{\dot H^s}.\end{aligned} \tag{7.29}$$

Proof. Similar to the previous lemma. □

Lemma 7.14. *Let $s \ge 0$. Then the nonlinear operator A_{res} defined in (3.21) maps $\dot H^s$ into $\dot H^{s+1}$ and satisfies the estimate*

$$\|A_{\text{res}}(v)\|_{\dot H^{s+1}} \le c_5(s)\|v\|_{\dot H^s}. \tag{7.30}$$

Proof. Clearly, (3.21) implies (7.30) with $c_5(s) \le \|v\|^2_{\dot H^0}$. If energy is conserved, then $c_5(s) \le \|v^0\|^2_{\dot H^0}$. □

Lemma 7.15. *Let $s > 1/2$. Then the trilinear operator R_3 defined in (3.17) maps $(\dot H^s)^3$ into $\dot H^s$ and satisfies the estimate*

$$\|R_3(u,v,w)\|_{\dot H^s} \le c_6(s)\|u\|_{\dot H^s}\|v\|_{\dot H^s}\|w\|_{\dot H^s}. \tag{7.31}$$

Proof. By (3.17)

$$|R_3(u,v,w)_k| \le \sum_{k_1+k_2+k_3=k} \frac{|u_{k_1}v_{k_2}w_{k_3}|}{|k_1|}. \tag{7.32}$$

As before, the time dependent exponentials in the definition of R_3 do not play a role. Setting $\tilde u_k = |u_k||k|^s$, $\tilde v_k = |v_k||k|^s$, $\tilde w_k = |w_k||k|^s$, $\tilde z_k = |z_k||k|^{-s}$ and arguing by duality we

have (taking no advantage of the factor $|k_1|^{-1}$ below)

$$|(R_3(u,v,w),z)| \le \sum_{k_1,k_2,k_3} \frac{\tilde u_{k_1}\tilde v_{k_2}\tilde w_{k_3}\, \tilde z_{k_1+k_2+k_3}|k_1+k_2+k_3|^s}{|k_1|^{s+1}|k_2|^s|k_3|^s} \le$$

$$3^{s-1}\sum_{k_1,k_2,k_3} |\tilde u_{k_1}\tilde v_{k_2}\tilde w_{k_3}\, \tilde z_{k_1+k_2+k_3}| \left(\frac{1}{|k_1|^s|k_2|^s}+\frac{1}{|k_1|^s|k_3|^s}+\frac{1}{|k_2|^s|k_3|^s}\right) \le$$

$$3^s c(s)^2 \|u\|_{\dot H^s}\|v\|_{\dot H^s}\|w\|_{\dot H^s}\|z\|_{\dot H^{-s}}\,.$$

where $c(s) = (\sum_{j\in\mathbb{Z}_0} j^{-(2s+2)})^{1/2} < \infty$ for $s > 1/2$. This proves (7.31) with $c_6(s) = 3^s c(s)^2$.

We finally observe that R_3 is a Lipschitz map from $\dot H^s$ to $\dot H^s$:

$$\|R_3(u_1,u_1,u_1) - R_3(u_2,u_2,u_2)\|_{\dot H^s} \le 10c_6(s)(\|u_1\|^2_{\dot H^s} + \|u_2\|^2_{\dot H^s})\|u_1-u_2\|_{\dot H^s}. \tag{7.33}$$

□

Lemma 7.16. *Let $S > 1/2$ and $\beta < 1/2$. Then the trilinear operator R_3 defined in (3.17) satisfies the estimate*

$$\|R_3(u,v,w)\|_{\dot H^{-S}} \le c_6'(S,\beta)\|u\|_{\dot H^{-\beta}}\|v\|_{\dot H^0}\|w\|_{\dot H^0}. \tag{7.34}$$

Proof. Arguing by duality and using (7.32) and (7.9) we obtain

$$|(R_3(u,v,w),z)| \le \sum_{k_1,k_2,k_3} \frac{u_{k_1}v_{k_2}w_{k_3}z_{k_1+k_2+k_3}}{|k_1|} \le$$

$$\left(\sum_{k_1,k_2,k_3} \frac{|u_{k_1}|\,|v_{k_2}|^2\,|z_{k_1+k_2+k_3}|}{|k_1|}\right)^{1/2} \left(\sum_{k_1,k_2,k_3} \frac{|u_{k_1}|\,|w_{k_3}|^2\,|z_{k_1+k_2+k_3}|}{|k_1|}\right)^{1/2} =$$

$$\left(\sum_{k_1}\frac{|u_{k_1}|}{|k_1|}\sum_{k_2}|v_{k_2}|^2\sum_{k}|z_k|\right)^{1/2} \left(\sum_{k_1}\frac{|u_{k_1}|}{|k_1|}\sum_{k_2}|w_{k_2}|^2\sum_{k}|z_k|\right)^{1/2} \le$$

$$c_2'(S)\|v\|_{\dot H^0}\|w\|_{\dot H^0}\|z\|_{\dot H^S}\left(\sum_{k_1}\frac{|u_{k_1}|^2}{|k_1|^{2\beta}}\right)^{1/2}\left(\sum_{k_1}\frac{1}{|k_1|^{2-2\beta}}\right)^{1/2}.$$

This proves (7.34). □

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